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Abstract

Larger firms (by sales or employment) have higher leverage. This pattern is explained using a model in which firms produce multiple varieties and borrow with the option to default against their future cash flow. A variety can die with a constant probability, implying that bigger firms (those with more varieties) have a lower coefficient of variation of sales and higher leverage. A lower risk-free rate benefits bigger firms more as they are able to lever more and existing firms buy more of the new varieties arriving into the economy. This leads to lower startup rates and greater concentration of sales.

Keywords: Startup rates, concentration, leverage, firm dynamics

JEL Codes: E22 E43 E44 G32 G33 G34

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1 New Introduction

Across a range of advanced economies, firm leverage is increasing in firm size. Rajan and Zingales (1995) documented this positive relationship for publicly traded firms for several OECD countries, including the U.S. Recently, Dinlersoz, Kalemli-Ozcan, Hyatt, and Penciakova (2019) have shown that the positive relationship between leverage and firm size also extends to private U.S. firms. Extant models of firm leverage, however, do not explain this fact. In macroeconomics, the most well-known model of firm leverage, Cooley and Quadrini (2001), predicts a negative relationship between size and leverage.[1] In finance, canonical models of firm capital structure (Leland (1994), Leland and Toft (1996)) are solved under assumptions that imply that optimal leverage is constant and independent of firm size.[2]

The main goal of this paper is to propose an explanation for the positive relationship between leverage and firm size in a model in which the growth of existing firms, as well as the entry of new firms, is endogenous. In our model, the growth of the business sector is driven by the steady arrival of ideas for new product varieties. To endogenize firm growth and firm entry, we assume that each idea for a new variety is initially owned by someone and the owner can either sell the idea to an existing firm, leading to the growth in output of that firm, or use the idea in a startup, leading to the entry of a new firm.

To get at the positive relationship between firm size and leverage, we assume that an existing product variety can go extinct with a constant probability, which implies that the growth rate of output of larger firms that manage a larger number of varieties is less volatile. Firms can borrow with the option to default on their debts. Since output growth of larger firms is less volatile, their likelihood of default on any given level of debt is lower and, under certain conditions, this implies that larger firms will choose to be more leveraged.

Our theory is novel in the way in which it explains the positive relationship between firm size and firm leverage. A striking feature of our theory is its novel implications for the connection between the market value of debt, the startup rate and business concentration. As we show later in the paper, both analytically (in a stripped-down version of the model) and numerically, a decrease

[1] In their model, growth stems from capital accumulation and finance is needed for investment. In this framework, diminishing returns to capital imply a negative – not positive – relationship between firm size and leverage.

[2] Of course, it is the case that optimal leverage depends negatively on the volatility of the firm's fundamental cash flow. More to the point, optimal capital structure models do not typically address the fact that businesses *choose* their size (Miao (2005) is an exception).

in the risk-free rate increases the *financial synergy* from a firm's acquisition of a new variety, and does so *more* for larger firms because they are more leveraged. Thus, a decline in the risk-free rate results in more new varieties being bought by larger firms, which leads simultaneously to a decline in the startup rate and to a rise in business concentration (share of sales accounted for by larger firms). Since the risk-free rate has in fact been declining for at least two decades, our model raises the possibility that the concomitant rise in business concentration and declines in startup rates — phenomena that have received a great deal of separate attention — may have a common cause in falling interest rates.

Our model is deliberately bare bones, but its key elements are grounded in well-established facts. The model relies on the volatility of growth of output being lower for larger firms, for which there is clear evidence. For the U.S., Davis, Haltiwanger, Jarmin, Krizan, Javier, Nucci, and Sandusky (2007, Figure 12) document that volatility in revenue growth declines with firm size. These authors take into account the contribution of exit to volatility in firm output. In an earlier study, Stanley, Amaral, Buldyrev, Havlin, Leschhorn, Maass, Salinger, and Stanley (1996) document that among publicly traded manufacturing firms that survive from one period to the next (so exit is ignored), the standard deviation of the growth rate of sales falls with firm size.[3]

The fact that larger businesses have less volatile growth rates does not automatically imply that they will have higher leverage. For this, it has to be easier for larger firms to borrow. We model this in two ways. In the main text, we assume that when firms borrow, they must obey a default probability constraint. The motivation for this constraint is the well-established fact that risky borrowers are simply denied credit.[4] Later in the paper (Section 6), we explore the case where there is no constraint on default probability but default imposes a fixed cost on creditors. Since volatility of the growth rate of cash flow falls with firm size, either approach generates a positive association between leverage and firm size.

Our model embeds a theory of firm entry that recognizes that ideas for new products occur to *people* and they get to choose the organizational form in which to implement them (Chatterjee

[3]Relatedly, Decker, D'Erasmo, and Boedo (2016) show that diversification — measured as the number of markets a firm is exposed to — is pro-cyclical and it is the larger firms that respond more in this way.

[4]The most direct evidence on this comes from surveys of small business lending: Federal Reserve Bank of New York (2017, p. 19) reports that among firms aged less than 5 years, 69 percent of those that requested credit received less credit than they sought; this percentage rises to 85 among businesses classified as medium/high risk. And even for investment banks, reputational concerns constrain the riskiness of the bonds they underwrite (Fang (2005)).

and Rossi-Hansberg (2012), Zábojník (2019)).[5] The theory applies when a person is contemplating setting up a pizza store and chooses between proceeding independently or as a franchise of a nationally-known brand. Similarly, the founders of spinoffs — new ventures that originate out of an existing one and is in the same line of business as its parent — often make a such a choice.[6] The other side of the coin, of course, are the many instances in which employees with new ideas implement their ideas in the firm in which they work. One indication that this occurs is the common practice of patent assignment, which transfers patent (or patent application) rights from inventor-employees to their employers. Another indication is the fact that, as assumed in this paper, individuals with valuable knowledge/ideas are compensated in the form of equity claims to future cash flows (Eisfeldt, Falato, and Xiaolan (undated)).

Our model is abstract in that firm growth is modeled as occuring only via the addition of new product lines of a fixed scale, and not also through increases in the scale of particular product lines.[7] While it is clear that stability of sales growth improves with size, not much is known about why this is the case. But large firms are generally viewed as having more diversified sources of revenue and "growth by new product lines" is a simple way to model this view. In addition, this approach establishes a bridge to theories of endogenous growth (Romer (1990), Grossman and Helpman (1991), Aghion and Howitt (1992)) in which growth occurs via the arrival and absorption of new goods (varieties). While we don't model the production of new ideas and goods, we consider a different aspect: Once the idea for a new good comes about, is it implemented in a startup or in an existing business and how is this choice affected by the risk-free rate? The choice has implications for the distribution of output across *firms* (business concentration), an important aspect of growth.[8]

Turning to the decline in the risk-free rate, the view that emerges from the many studies that have examined its possible causes (Caballero, Farhi, and Gourinchas (2008), Mendoza, Quadrini, and Ríos-Rull (2009), Eichengreen (2015), Del Negro, Giannone, Giannoni, and Tambalotti (2017),

[5]In both these earlier papers, the choice depended on the quality of the idea, with lower quality ideas being sold and higher quality ideas leading to startups. In contrast, in this paper the choice depends on the technological and financial synergies between the idea and existing firms.

[6]For example, Klepper (2007) argues that startups in the early auto industry resulted from disagreements among employees about the viability of new ideas, which suggests that the new ideas *could* have been implemented in the parent firm but the idea's owner chose not to do so.

[7]By ignoring growth via scale of production, we also abstract from the connection between leverage and the intensive margin of production that has been the focus of the many papers that have followed in the wake of Cooley and Quadrini (2001). Studies that focus on firm investment in the presence of default risk or borrowing constraints include Arellano, Bai, and Zhang (2012), Jermann and Quadrini (2012), Khan and Thomas (2013), Arellano, Bai, and Kehoe (2016), Gomes, Jermann, and Schmid (2016) and Corbae and D'Erasmo (2017), among others.

[8]See Luttmer (2010, Section 3.4, p. 559) and Chatterjee and Rossi-Hansberg (2012) for an in-depth discussion of this point.

Farhi and Gourio (2018), among others) is that it has resulted largely from a rise in the premium placed on safety and liquidity. Consistent with this, our model treats the decline in the risk-free rate as occurring due to a change in the preferences of lenders. Since we assume that lenders are risk-neutral, the decline is modeled simply as a decline in their degree of impatience.

The causes of the decline in the startup rate — often described as a "decline in business dynamism" — and the rise in business concentration are active areas of research. Regarding the decline in the startup rate, Hathaway and Litan (2014) list several factors, including slowing population growth, increasing business consolidation, and the rising burden of regulation and taxes as potential causes. The role of slowing of labor force growth has been stressed in Karahan, Pugsley, and Şahin (2019) and in Hopenhayn, Niera, and Singhania (2018), and that of changes in corporate tax rates in Neira and Singhania (2017). Studies that attempt to explain the rise in business concentration have examined increases in market power (De Loecker and Eeckhout (2017)) and the entry of large firms into new geographic markets (Hsieh and Rossi-Hansberg (2020), Rossi-Hansberg, Sarte, and Trachter (forthcoming)).

In recent work, Aghion, Bergeaud, Boppart, Klenow, and Li (2019) and Akcigit and Ates (2019) explore, like us, a common cause for the decline in entry rates and the rise in concentration. The former focus on technological change that is increasingly benefitting larger firms and the latter on a decline in knowledge diffusion from leading to lagging firms. Liu, Mian, and Sufi (2019) explore the role of low interest rates in generating greater business concentration through a "strategic competition effect" but the connection to lower entry rates is not made. Although the conceptual frameworks of these studies are quite different from ours, they all share the key commonality that new ideas/products are increasingly appearing within larger firms.

The paper is organized as follows. Section 2 lays out our model of firm dynamics with borrowing, default, entry and exit. In Section 3, a simple stripped-down version of this model is analyzed to explain the key idea of this paper: the market value of debt can affect the entry rate of new firms and the growth rate of existing firms. Section 4 analyzes the full model numerically and establishes that key results derived in the stripped-down model carry over to the full model for realistic parameter values. Section 5 analyzes the impact of a decline in the risk-free rate on firm dynamics and shows that the numerical model is capable of accounting for the magnitude of the decline in the entry rate since the late 1990s. Section 6 describes a model in which creditors incur a fixed default cost in the event of bankruptcy and shows that this alternative setup has similar properties as our main model. Section 7 concludes.

2 Model

Time is discrete and the economy is composed of a continuum of heterogeneous firms. At the start of a period, the state of a firm is the pair (K, B), where K is the number of varieties owned by the firm and B is its debt. We assume there is an upper bound $K_{\max}$ to the number of varieties that a single firm can manage, so $K \in \mathbb{K} \equiv \{1, 2, 3, \ldots, K_{\max}\}$. We assume that there is an upper bound $B_{\max}$ on the amount of debt a firm can carry and, so, $B \in [0, B_{\max}]$. It will turn out that there is an endogenous upper bound on the amount of debt that can be issued and, so, if $B_{\max}$ is sufficiently large, it will be nonbinding. Also, in our theory (as well as in our numerical simulations), firms will have an incentive to issue debt rather than save and, so, to keep the notation streamlined, the option to save is removed from the choice set.

To minimize technicalities, we approximate the interval $[0, B_{\max}]$ by a finite set (discrete approximation) $\mathbb{B}$. Then, the aggregate state of the economy is nonnegative vector $\{\mu(K, B),\ (K, B) \in \mathbb{K} \times \mathbb{B}\}$ where $\mu(K, B) \geq 0$ is the mass of firms in state (K, B).

2.1 Arrival of New Ideas and Their Implementation

Each period, ideas for new varieties arrive in the economy. We assume that ideas occur to workers. The measure of workers in the economy as a whole is taken as fixed. The measure of new ideas arriving into the economy each period is also fixed and given by $M > 0$.

A key aspect of our model is the assumption regarding how knowledge of these newly arriving ideas are distributed across firms in the economy. We assume that workers in a firm that owns K varieties generate ideas for new varieties at the rate ρK. Here ρ – the rate per variety – is taken as given by all decision makers, but its value is determined in equilibrium. The proportional relationship between the number of ideas for new varieties generated and K reflects our background assumption that employment is proportional to K.

To keep the exposition streamlined, in the main body of the paper we assume that the period is short enough, and so ρ small enough, so that $\rho K_{\max} < 1$. Then, if a firm gets an opportunity to purchase an idea, it knows that this is the only such opportunity it will get this period. In the numerical analysis, the period length is fixed at a month and ρ is determined by the calibration of the model and sufficiently big firms get multiple ideas in a month.[9]

To incorporate a choice between selling an idea to an existing firm and implementing the idea in a startup we assume that an idea is potentially valuable to the firm in which it arose. Specifically, we assume that an idea can be successfully turned into a new variety by the firm in which it arose with probability $s \in [0, 1]$. For each idea, the value of s is drawn independently from a distribution $F(s)$. A higher s indicates greater *technological synergy* between the idea and the firm's existing capabilities. If the idea is implemented in a startup, we assume that it is successfully transformed into a new variety with constant probability σ.

Whether the idea is implemented in the firm or in a startup is determined via bargaining between the firm and the worker who got the idea.[10] Let $W(N, B)$ denote the value of a firm that owns N varieties and has debt B after all opportunities for purchase of ideas have been exhausted. If the firm is not at capacity (i.e., $K < K_{\max}$), there is a potential gain from implementing the idea in the firm if

$$s[W(K+1, B) - W(K, B)] \geq \sigma W(1, 0). \tag{1}$$

The l.h.s. is the expected gain to the firm if it implements the idea for a new variety. The r.h.s. is the expected gain to a startup which by definition owns one variety and has no pre-existing debt. If the surplus is nonnegative, we assume that the new variety is implemented by the firm and the inventor gets $\sigma W(0, 1)$ through shares in the existing firm.[11,12]

Let $s^*(K, B)$ solve (1) with equality.[13] If $K < K_{\max}$, the idea is implemented in the firm if $s \geq s^*(K, B)$ and in a startup if $s < s^*(K, B)$. If $K = K_{\max}$, the idea is implemented in a startup

[9] When $\rho K_{\max} > 1$, it is necessary to introduce an integer function $N(K)$ such that $N(K) + 1 > \rho K \geq N(K)$. Then our assumption is that workers in a firm that owns K varieties generate $N(K)$ ideas for sure and generate the $N(K) + 1$ th idea with probability $\rho K - N(K)$. The decision problem corresponding to this general case is stated in Appendix A.

[10] In the general case where multiple workers in a firm could receive ideas, we assume that the bargaining proceeds sequentially: The firm bargains with a randomly chosen worker with an idea knowing only the probability distribution of how many more workers are in line to bargain and knowing whether previously purchased ideas have been successfully turned into new varieties or not. See Appendix B for the firm's Bellman equations for this case.

[11] If the firm has unit shares outstanding, it issues a additional units to the inventor, where $a = \sigma W(1, 0)$. The value of $1 + a$ units of shares is $s[W(K+1, B) - W(K, B)] + W(K, B)$ and the post-purchase value of the original unit share is $s[W(K+1, B) - W(K, B)] + W(K, B) - \sigma W(1, 0)$.

[12] If, instead, we assumed that inventors must be compensated in cash, firms would have an incentive to accumulate cash balances to purchase ideas (fund investment opportunities), which we do see firms doing. We note, however, that it is possible for a 1-variety firm to not choose to accumulate any cash (if the interest rate is too low or investment opportunities arrive too rarely). In that case, the long-run equilibrium would feature only 1-variety firms and firms would not grow over time.

[13] Note that $s^*(K, B)$ can exceed 1 or fall below 0.

regardless of the value of s. Under these assumptions, the start-of-the-period value of the firm, denoted $Z(K, B)$, is

$$Z(K,B) = \begin{cases} W(K,B) & \text{if } K = K_{\max} \\ \rho K \int \mathbb{1}_{\{s \geq s^*(K,B)\}} \left[sW(K+1,B) + (1-s)W(K,B) - \sigma W(1,0) \right] dF(s) + & \\ \qquad [1 - \rho K \int \mathbb{1}_{\{s \geq s^*(K,B)\}} dF(s)] W(K,B) & \text{if } K < K_{\max}. \end{cases}$$

2.2 Destruction of Varieties

Let N denote the number of varieties owned by the firm, including any that it bought in the current period and successfully integrated into its portfolio. At this juncture, each variety in existence can become extinct with probability ϕ. The probability that a firm with N varieties ends up with $0 \leq K' \leq N$ varieties is, therefore,

$$x(N, K') = \binom{N}{K'} (1-\phi)^{K'} \phi^{N-K'}.$$

2.3 Debt, Default and Exit

Following the extinction shocks, the output of the firm is realized. We assume that each variety generates a cash flow (revenue less the costs of production) of $\pi > 0$ and, so, the total cash flow of the firm is $\pi K'$. At this point, the firm services any existing debt if it can and potentially engages in new debt issuances. If the firm issues new debt B', the price at which it sells the debt is $q(K', B')$. Both K and B are relevant for assessing the probability of default on debt and, so, both appear as arguments of the bond price.

We impose two important constraints on a firm's debt decisions. First, we rule out equity infusions, i.e., B' must respect nonnegative dividend payouts:

$$\pi K' - B + q(K', B')B' \geq 0. \tag{2}$$

If the firm's cash $\pi K'$ falls short of its obligations B, its only option is to meet the deficit by debt issuance. Effectively, the constraint imposes a lower bound on B'. And, second, B' must respect a default probability constraint:

$$d(K', B') \leq \theta, \ \ \theta \in (0, 1], \tag{3}$$

where $d(K', B')$ is the probability of default on debt issued in the current period. As mentioned in the introduction, the motivation for this constraint is the well-established fact that lenders deny credit to risky firms. In Section 6 we consider the alternative case where default imposes a fixed cost on lenders.

A firm is in default if it cannot meet its debt obligations fully. To formalize this, let $G(K')$ be the highest revenue from bond sales consistent with the bonds meeting the default probability constraint (3). That is,

$$G(K') = \max_{B'} q(K', B')B' \tag{4}$$

s.t.

$$d(K', B') \leq \theta.$$

Define $\bar{B}(K') = \pi K' + G(K')$. Then default will occur if $B > \bar{B}(K')$ since the sum of its cash flow and the maximum amount of external finance available to it falls short of its debt obligation.

Since default occurs only when repayment is impossible, the default decision rule $D(K', B)$ is mechanically given by

$$D(K', B) = \begin{cases} 1 & \text{if } B > \bar{B}(K') \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

In the event of default, the debt owed to creditors is reduced to $\bar{B}(K')$. The value of a firm in default is then given by:

$$V^D(K', B) = \max_{B'} \pi K' - \bar{B}(K') + q(K', B')B' + \beta Z(K', B')$$

s.t.

$$q(K', B')B' = G(K')$$

$$d(K', B') \leq \theta.$$

Here $0 < \beta < 1$ is the discount factor of owners and workers. If there is a unique B' that attains $G(K')$, it is also the B' that (trivially) solves the firm's optimization problem under default: it is the only choice that is available to the firm. The dividend payout in default is zero but owners retain rights over the firm's future cash flow. Thus, default resembles a Chapter 13 corporate reorganization rather than an outright business liquidation.

If the firm does not default, that is $B \leq \bar{B}(K')$, the firm solves

$$\begin{aligned}
&V^R(K', B) = \max_{B'} \pi K' - B + q(K', B')B' + \beta Z(K', B') \\
&\text{s.t.} \\
&\pi K' - B + q(K', B')B' \geq 0 \\
&d(K', B') \leq \theta.
\end{aligned}$$

Exit of a firm occurs if it loses all its varieties. This is because if $K' = 0$, the firm cannot acquire new varieties (since $\rho \times 0 = 0$) and 0 becomes an absorbing state. At $K' = 0$, the probability of default on any amount of debt is 1 (as K'' is zero with certainty) and, so, $\bar{B}(0) = 0$. Hence, if $B > 0$, the exiting firm defaults and creditors get nothing.

We can now give the expression for the probability of default on bonds issued in the current period:

$$\begin{aligned}
d(K', B') = [\rho K' \int s \mathbb{1}_{\{s \geq s^*(K',B')\}} dF(s)] \mathbb{E}_{(K''|K'+1)} D(K'', B') + \\
[1 - \rho K' \int s \mathbb{1}_{\{s \geq s^*(K',B')\}} dF(s)] \mathbb{E}_{(K''|K')} D(K'', B').
\end{aligned} \tag{6}$$

The first term is the probability that the firm successfully adds to its product portfolio next period multiplied by the probability of default conditional on having a new variety.[14] The second term is an analogous term covering the complementary case where the firm fails to acquire a new variety

[14]The full expression for the probability of default conditional on acquiring new variety next period is

$$\sum_{K''=0}^{K'+1} x(K'+1, K'') \mathbb{1}_{\{(B' > \bar{B}(K''))\}},$$

which is equivalent to $\sum_{K''=0}^{K'+1} x(K'+1, K'') D(K'', B')$ (similarly for the probability of default conditional on not acquiring a variety).

next period (either because no ideas were generated in the firm or because the generated idea had too low an s).

Finally, we give the expression for $W(N,B)$:

$$W(N,B) = \mathbb{E}_{(K'|K'\leq N)}\left[[1-D(K',B)]V^R(K',B) + D(K',B)V^D(K',B)\right]. \tag{7}$$

2.4 Equilibrium Conditions

The first equilibrium condition pertains to the pricing of bonds. We assume that lenders are risk neutral and lending is a competitive business. The equilibrium condition implied by these assumptions is that the expected rate of return on a bond must equal the risk-free interest rate. This can then be expressed as:

$$\begin{aligned} &q(K',B')(1+r) \qquad (8)\\ &= \left[\rho K' \int s\mathbb{1}_{\{s\geq s^*(K',B')\}}dF(s)\right]\mathbb{E}_{(K''|K'+1)}\left[[1-D(K'',B')] + D(K'',B')\frac{\bar{B}(K'')}{B'}\right] + \\ &\quad \left[1-\rho K' \int s\mathbb{1}_{\{s\geq s^*(K',B')\}}dF(s)\right]\mathbb{E}_{(K''|K')}\left[[1-D(K'',B')] + D(K'',B')\frac{\bar{B}(K'')}{B'}\right]. \end{aligned}$$

The l.h.s. is the opportunity cost of investing in a unit bond with price $q(K',B')$, and the r.h.s. is the expected payoff from doing so. The first term on the r.h.s. is the product of the probability of acquiring a new variety and the expected payoff on the bond conditional on having added a new variety. The payoff is 1 if there is no default, and it is the repayment ratio $\bar{B}(K'')/B'$ if there is default. The second term is similar, covering the case where a new variety is not acquired.

The second equilibrium condition pertains to ρ. Its value must be consistent with the total measure of varieties in steady state. To express this condition, let $\mu^*(K,B;\rho)$ denote the steady-state measure of firms of type (K,B), given ρ. Then,

$$M = \rho \sum_{K\in\mathbb{K}} \sum_{B\in\mathbb{B}} K\mu^*(K,B;\rho). \tag{9}$$

This equation determines a ρ that makes the number of varieties constant and equal to M/ρ.[15] If M is changed by a factor λ, the value of ρ does not change but $\mu(K, B)$ changes by the same factor for all (K, B) and so does the number of varieties.

3 External Finance and Value of New Ideas

The goal of this section is to explain, in the context of the stripped-down version of the model, how access to external finance makes new ideas more valuable to larger firms.

Consider a version in which $K_{\max} = 2$. Thus, a firm can own one or two varieties. In addition, assume that there are only two periods: the current period and a final period. The timeline of events in the current period is exactly as described in the model. The final period is a "dummy" period in which only extinction shocks happen, followed by realization of output, debt payments and consumption of dividends.

In this two period world, default on debt will occur if and only if the cash flow in the final period is less than the debt owed.[16] That is, if and only if

$$\pi K'' < B',$$

where we are using K' to denote the number of varieties that survive in the current period and K'' the number that survives in the final period.

The default probability constraint is:

$$\Pr\left\{\pi K'' < B'\right\} < \theta. \tag{10}$$

The amount a firm can borrow depends on K', θ and ϕ. If $\theta < 1$, a firm can never borrow more than the maximum possible cash flow next period because if it did, it would default with certainty. Thus, for $\theta < 1, B' \leq \pi K'$. For $\theta = 1$, $B' > \pi K'$ is possible, but committing more than what the firm can ever pay simply leads to a reduction in prices with no change in revenue from bond sales. So, without any loss of generality, we may assume that $B' \leq \pi K'$, even if $\theta = 1$.

[15] For a given K, the inner sum gives the measure of varieties owned by all firms owning K varieties each, and summed over K, the outer sum gives the total measure of varieties.

[16] In the main model, this a necessary condition for default but it is not sufficient because firms can avoid default by issuing new debt to pay of existing creditors.

First, we will focus on the values of θ for which leverage is increasing in size, since that is the case most relevant with respect to the data and the quantitative results of the next section.

Proposition 1. *Suppose* $(1+r)\beta < 1$ *and* $\phi^2 < \theta < \phi$. *Then (i)* $B'(K') = (K'-1)\pi$, *(ii)* $s^*(1,0) < \sigma$, *(iii)* $s^*(1,0)$ *is increasing in* r, *(iv)* $s^*(1,B)$ *is increasing in* B.

Proof. (i). Consider a firm with $K' = 1$. If it borrows some amount $0 < B' \leq \pi$, it will default with probability ϕ. Since $\theta < \phi$, the default probability constraint is violated for any such B and the firm cannot borrow at all. So, $B'(1) = 0$.

Consider a firm with $K' = 2$. If the firm borrows $\pi < B' \leq 2\pi$, it will default if it loses one or both varieties next period. The probability of this event is $1-(1-\phi)^2$. For these levels of borrowing to be feasible, $1-(1-\phi)^2 < \theta$, which is impossible, given $\theta < \phi$. If the firm borrows $0 < B' \leq \pi$, it will default if it loses both varieties, the probability of which is ϕ^2. Since $\phi^2 < \theta$, these borrowing levels are feasible. What is the firm's optimal borrowing level in this range? If the firm borrows B', it gets $q(1,B')B'$ in the current period and promises to repay B' with probability $(1-\phi^2)$ next period. In equilibrium, $q(1,B') = (1-\phi^2)/(1+r)$, and so its net gain from issuing B' units of debt is $[(1-\phi^2)B'[(1+r)^{-1}-\beta]$. Since $\beta(1+r) < 1$, the net gain is maximized by committing as much of next period's output to lenders as possible. So, $B'(2) = \pi$.

(ii) $s^*(1,0)$ solves

$$s[W(2,0) - W(1,0)] = \sigma W(1,0).$$

Since a single variety survives with probability $(1-\phi)$,

$$W(1,0) = [(1-\phi)(\pi + \beta(1-\phi)\pi)].$$

The difference $W(2,0) - W(1,0)$, with some simplications, can be expressed as

$$\left[W(1,0) + (1-\phi)^2 \left(1-\phi^2\right)\pi\left(\frac{1}{(1+r)} - \beta\right)\right].$$

This expression is intuitive. The gain from acquiring a variety is composed of two parts. The first part is the value of expected output from the new variety, which is $W(1,0)$. But, in addition, if both varieties survive the extinction shocks in the current period, which occurs with probability

$(1-\phi)^2$, the firm gets to issue π units of debt. The term multiplying $(1-\phi)^2$ is the net utility gain from issuing $B' = \pi$. Therefore, $[W(2,0) - W(1,0)] > W(1,0)$ and $s^*(1,0) < \sigma$.

(iii). Since $W(1,0)$ is independent of r, it follows that $[W(2,0) - W(1,0)]$ is negatively related to r. Therefore, $s^*(1,0)$ is positively related to r.

(iv). Finally, consider the case where a 1-variety firm enters with some debt $B \leq \pi$. Then $W(1,0)$ is the same as before, but the gain from acquiring a new variety is

$$W(2,B) - W(1,0) = (1-\phi)(\pi - \phi B) + \beta (1-\phi)^2 \pi + (1-\phi)^2 \left(1-\phi^2\right) \pi \left(\frac{1}{(1+r)} - \beta\right).$$

Hence $s^*(1,B)$ is increasing in B. □

Proposition 2. *Suppose* $(1+r)\beta < 1$ *and* $\phi < \theta \leq 1$*. Then (i)* $B(K') = \pi K'$ *for* $K' \in \{1,2\}$ *and (ii)* $s^*(K',0) = \sigma$ *for all* r*.*

Proof. (i) If $\theta > \phi$, the 1-variety firm can borrow positive amounts and a 2-variety firm can now borrow more than π without violating the default constraint. Since $\beta(1+r) < 1$, a 1-variety firm will borrow π and 2-variety firm will borrow 2π. (ii) For these levels of borrowing, we can verify that $W(2,0) = 2W(1,0)$, regardless of the value of r. Hence $[W(2,0) - W(1,0)] = W(1,0)$ and $s^*(1,0) = \sigma$ for all r. □

Proposition 3. $0 \leq \theta < \phi^2$ *neither firm can borrow and* $s^*(1,0) = \sigma$

Proof. For these levels of θ, neither 1- nor 2-variety firms can borrow at all since default probability on any level of borrowing will exceed θ. In this case, one may verify that $W(2,0) = 2W(1,0)$ also, and so $[W(2,0] - W(1,0)] = W(1,0)$ and $s^*(1,0) = \sigma$. □

Propositions 1 and 2 together show that the access to external finance raises the value of new ideas to existing firms if *leverage* is increasing with firm size. In Proposition 2, the larger firm can borrow more but equilibrium leverage of both firms is the same (and equal to 1). Consequently, there is no financial benefit or synergy from purchasing a new idea and it purchases only those ideas for which there is technological synergy, i.e., for which it has a higher success probability than a startup.

But if leverage is increasing in size, there is a financial benefit from a purchase and the firm will purchase an idea even its success probability is *lower* than the success probability of that idea in a startup, i.e., $s^*(1,0) < \sigma$. From a social point of view, this is a *misallocation of resources*: expected total output is reduced if ideas are implemented in the organization in which its success probability is lower.

Furthermore, as part (iii) of Proposition 1 indicates, $s^*(1,0)$ increases with r. Thus, the startup rate is negatively affected by a lower risk-free rate. This effect arises not because equilibrium leverage is affected by r (it is not) but simply because a lower r increases financial benefit of incorporating the new idea in the firm. By the same token, a lower r increases misallocation of resources.

Finally, part (iv) of Proposition 1 indicates that the financial benefit conferred on larger firms by their higher equilibrium leverage is reduced with legacy debt. This is just a manifestation of the well-known *debt overhang problem*. Note that for the expression for $[W(2,B) - W(1,0)]$ differs from $[W(2,0) - W(1,0)]$ in that the term corresponding to $W(1,0)$ is now $(1-\phi)(\pi - \phi B) + \beta(1-\phi)^2 \pi < W(1,0)$. Intuitively speaking, the subtraction term $-\phi(1-\phi)B$ can be explained as follows: In the event its own variety fails but the one it acquired survives — which happens with probability $\phi(1-\phi)$ — the firm must repay its debt. In other words, in the presence of legacy debt, a part of the expected increase in output from a new acquisition goes not to the firm but to its existing creditors, thus blunting the firm's incentive to acquire a new variety.

In this simple setup as well as in the full dynamic model, the default constraint plays an important role. However, we show in Section 6 that the positive relationship between size and leverage can also arise in a setup where there is no constraint on default probabilities but lenders incur a fixed cost in the event of default. In this alternative setup, small firms are, again, more constrained in their borrowing than large firms.

4 Numerical Analysis

We now turn to a numerical analysis of the full model. We set the model period to be a month and assign a realistic set of values to model parameters. The main objective is to explore how the steady state of the model varies with r.

Table 1:
Parameters Set Independently

Parameter, Annualized	Value
r	0.022
β	0.950
M	120
$K_{\max}$	65

The model has (i) two market parameters, r and θ, (ii) one preference parameter, β, (iii) three technological parameters, M, ϕ, and σ and (iv) the distribution $F(s)$. We take the distribution F to be uniform over the interval $[s_{\min}, 1]$, where $s_{\min} \geq 0$. With this distributional assumption, seven parameters need to be assigned.

The parameters listed in Table 1 are set independently. All values are reported in annualized form. The risk-free interest rate r is set to 2.16 percent per annum, which is the trend value of the annual average real return on 3-month Treasury bills in 1997.[17] The exact value of β is not important for the results, but it is important that firms be more impatient than lenders. We set β to 0.95. As noted earlier, the value of M is a normalization, and we set it to a numerically convenient value of 120 (i.e., 10 per month). The value for $K_{\max}$ was set at the smallest value consistent with model statistics being unaffected if the value is raised.

The top panel of Table 2 reports the remaining parameters, which are set jointly to deliver realistic model statistics. Each row lists the statistic that is matched and the parameter value that is most directly determined by the match.

The first statistic listed is the default probability on debt. For this statistic, we use the bankruptcy rate for firms reported in Corbae and D'Erasmo (2017, Table 1).[18] In the model, the parameter that most affects the default rate is θ and, so, this is the parameter that is listed for this row. Its value must be 0.05.

[17]The trend value in 1997 is the predicted value of a linear time trend regression over the period $1997 - 2015$.

[18]They report an overall bankruptcy rate of 0.96 percent for the period $1980 - 2014$. Once we take into account that only around 90 percent of firms carry debt, bankruptcy rate conditional on debt is 1.1 percent. The historical default rate on corporate bonds for the period $1983 - 2017$ reported in Moody's is 1.6 percent.

Table 2

Parameters Set Jointly			
Description of Statistic	Data	Parameter	Value
Probability of default	0.01	θ	0.050
Annual entry rate of new firms	0.11	σ	0.910
Survival rate of 1-yr-old firms	0.84	ϕ	0.195
Response of entry to r	-	$s_{\min}$	0.870
Equilibrium value	-	ρ	0.209

Notes: The data for entry rate of new firms and survival rate of existing firms are authors calculations based on data from the U.S. Census Bureau's Business Dynamics Statistics database (https://www.census.gov/ces/dataproducts/bds/data.html).

The next two statistics are the annual rate of entry of new firms and the survival rate of 1-year-old firms in 1997.[19] The parameters that most affect the entry rate are σ (the success probability of startups) and $s_{\min}$ (the lower bound of the support of s), and the entry rate is increasing in both σ and $s_{\min}$. As we discuss later in the paper, the value of $s_{\min}$ also affects the *response* of entry rates to decline in r. Anticipating that discussion, we set the value of $s_{\min}$ to 0.87. With this choice, the value of σ that matches the entry rate of firms is 0.91. The parameter that most affects the survival rate is ϕ (the product extinction probability), with the survival rate declining in ϕ. The implied value of ϕ is 0.195.

The last parameter listed in the table is the one whose value is determined in equilibrium, namely, ρ. If every idea were to be successfully implemented, then ρ would be equal to ϕ. But this is not the case and, so, in equilibrium, ρ must exceed ϕ. However, the parameters selected imply that more than 90 percent of the ideas arriving into the economy are successfully implemented, so ρ is only slightly greater than ϕ. We note that at the equilibrium monthly value of ρ, firms with $52 < K \leq K_{\max}$ get one idea for sure and a second idea with some probability (see Appendix A for a description of the optimization problem of a firm for the case where $\rho K > 1$.)

We now turn to properties of the model.

Leverage and Firm Size:

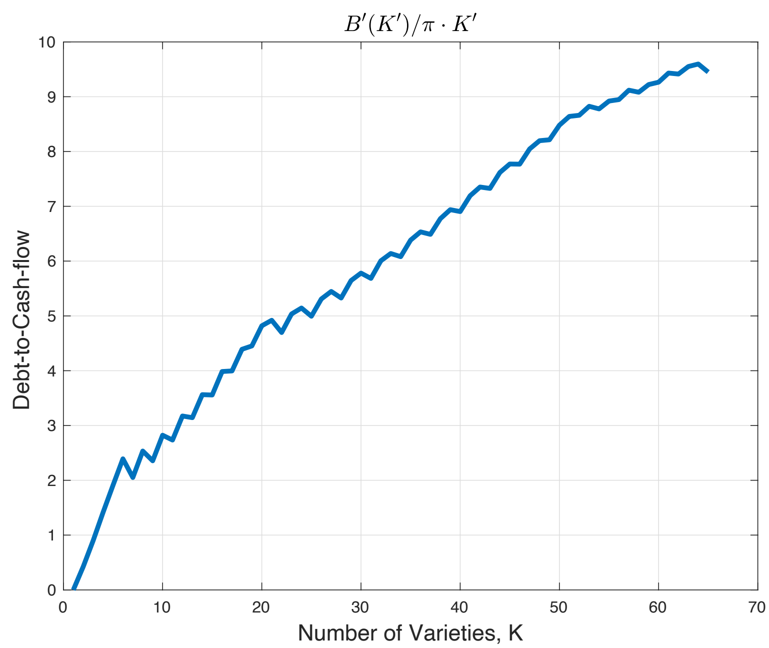
The property that is of most interest to us is the relationship between leverage and firm size. Figure

[19] As in the case of the real interest rate target, both rates are set to their respective trend values, as implied by linear trend regressions over the period $1997 - 2015$.

1 displays $[B'(K')/\pi K']$ against K'. Although the relationship is not strictly monotonic (more on this below), leverage is generally strongly increasing with firm size: A firm that owns a larger number of varieties is generally able (and willing) to borrow a greater multiple of its current-period cash flow.

As explained in the previous section, for the lenders, the risk associated with leverage depends on the number of varieties surviving next period as a proportion of the number of varieties today. Ignoring for the moment the possibility of acquiring a new variety, a firm will, on average, have roughly $(1-\phi)$ of its current varieties next period, and the variation around this proportion, as measured by the variance, is $[(1-\phi)\phi/N]$. Thus, the riskiness of the cash flow shrinks with N. Consequently, a bigger firm is able to borrow a higher proportion of its current-period cash flow before running into the default probability constraint.

Figure 1:
Leverage and Firm Size



Several remarks are worth making. First, the response of leverage to the logarithm of firm size predicted by the numerical model is about what it is in the data. Dinlersoz, Kalemli-Ozcan, Hyatt, and Penciakova (2019, Table 4) report that the response of the ratio of leverage (defined as the ratio of total financial debt to total assets) to the logarithm of firm employment is 0.0178 and 0.0281 among publicly listed and privately firms, respectively. In our model, the response of the leverage (defined as the ratio of debt to cash flow) to the logarithm of the number of varieties is 0.025.

Second, because of impatience, all firms (in the steady state equilibrium) are at their default constraint. Since θ is chosen to match a default rate of 0.01, we might expect θ to be 0.01, but its value, instead, is 0.05. This result is a consequence of $d(K', B')$ being a step function in B'. Because the probability of default jumps up discretely with increasing debt, there is no B' (generically) for which $d(K', B')$ is *exactly* 0.05 for any K'. Consequently, the equilibrium default probability for any firm is always strictly less than 0.05 and so is the average default probability.

Third, in the steady state equilibrium, the default probability is strictly decreasing in K'. This permits a firm with $K' + 1$ varieties to borrow more than a firm with K' varieties, i.e., $B'(K' + 1) > B'(K')$. But the additional borrowing can be quite modest, and in these instances, the $K' + 1$-variety firm's leverage, $B'(K' + 1)/\pi[K' + 1]$, can be less than $B'(K')/\pi K'$, the leverage of the K' variety firm. Of course, with a big enough increase in size, the firm is able (and willing) to increase its leverage (as shown in the figure).

Larger Firms Are More Willing to Buy New Ideas

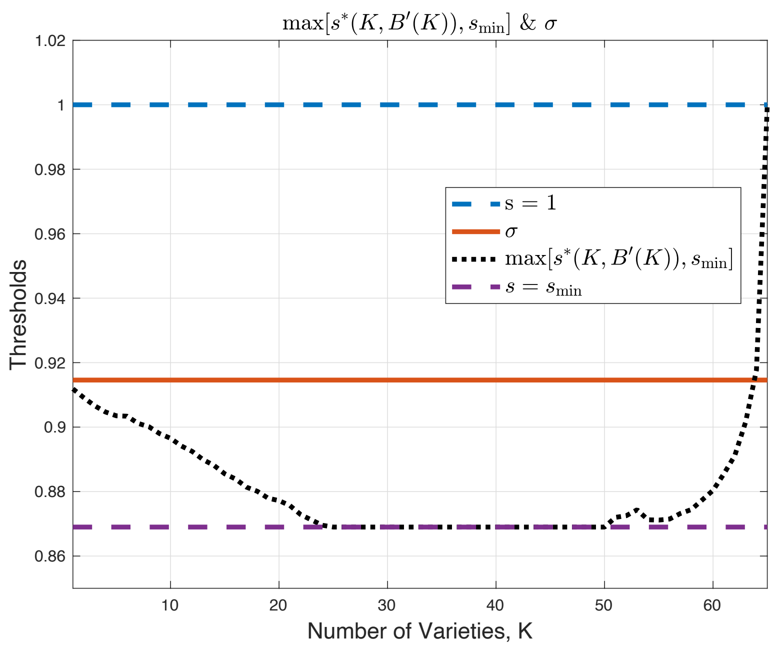
The message of Figure 1 is that larger firms are able to borrow a greater fraction of their cash flow. From our discussion in the context of the two period model, we should expect new ideas to be more valuable to larger firms. This point is generally confirmed in Figure 2 for the full model.

The dotted line in the figure plots the equilibrium threshold higher of $s^*(K', B'(K'))$ and $s_{\min}$ against K'. Along with this line, the figure plots three other horizontal lines. The top-most and bottom-most horizontal lines are the upper and lower bounds, respectively, of the support of s, and the middle horizontal line is drawn at the level of σ (the success probability of an idea in a startup).

As expected, the threshold declines monotonically with K', reaching $s_{\min}$ at the value of $K' =$ 24. Between firm size of 25 and 50, all ideas generated in the firm are implemented by it. Beyond firm size of 50, however, the upper bound on the number of varieties a firm can own, namely, $K_{\max}$, begins to exert an effect. The threshold is 1 at $K' = 65$, which is $K_{\max}$, because a new idea cannot be implemented in the firm and so there is no value to acquiring it. Because at $K_{\max}$ the firm must reject all ideas, no matter how high the success probability, it becomes more choosy as its size approaches $K_{\max}$ so as to not miss out on the opportunity to buy ideas with high success probabilities.

A couple of implications follow: First, up to a firm size of 50, a larger firm is at least as likely to buy new ideas than a smaller firm. Thus bigger firms will spawn *fewer* startups per worker than smaller firms, a pattern that is consistent with evidence reported in Elfenbein, Hamilton, and Zenger (2010). Second, for a given K', if s falls in the region between the solid red and the black dotted lines, there is misallocation: The idea is implemented in the firm where the success probability is lower than in a startup. There is a possibility of misallocation of this type for all firm sizes except the top two. And, up until $K' = 50$, the possibility of misallocation is at least as high for larger firms as it is for smaller ones. For the top two firm sizes there is also a possibility of misallocation but in the other direction: A firm may get an opportunity to buy an idea that has a higher success probability if implemented in the firm rather than a startup, but it will reject it because of an approaching or binding constraint on firm size.

Figure 2:
Firm Size and the Willingness to Buy New Ideas

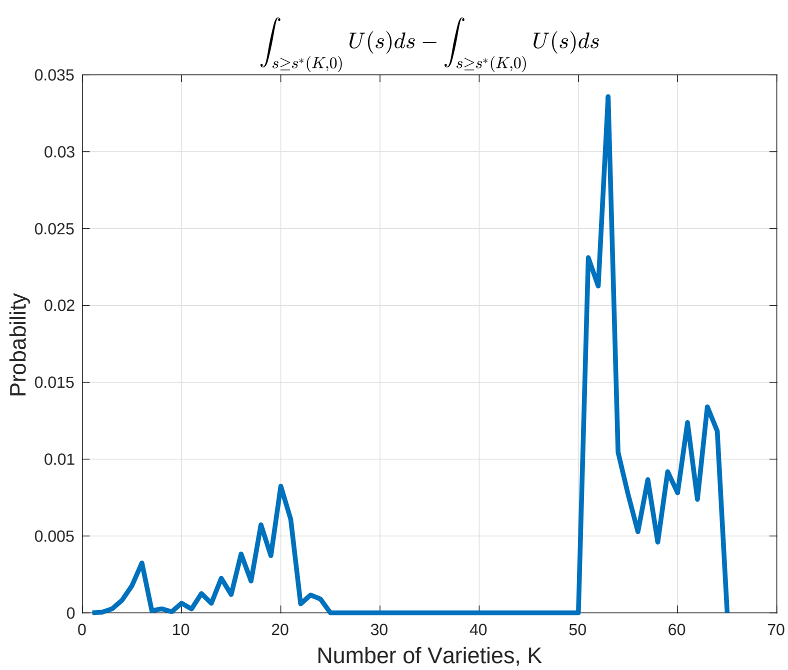


Debt Overhang

In the simple two period model of Section 3, we showed that legacy debt made the firm more choosy about ideas it buys. Figure 3 confirms this debt overhang effect in the full model. It plots the difference between $\int_{s \geq s^*(K',0)} U(s)ds$ and $\int_{s \geq s^*(K,B'(K'))} U(s)ds$, where $U(s)$ is the density of a uniform distribution with support $[s_{\min}, 1]$. In other words, it plots the increase in the probability of acceptance that would occur if a firm of size K arrived into the period with zero debt instead of the equilibrium level of debt. Since a firm with $K = 1$ cannot borrow, $s^*(1, B'(1) = s^*(1, 0)$. The gap is positive for firm sizes between 2 and 24. In the range between 25 and 50, the value of s^* is below $s_{\min}$ with or without debt and the presence of debt does raise the acceptance threshold. Between 51 and 64 the gap is positive again and relatively substantial.

The intuition for this debt overhang effect is similar to the logic of the two period model. Given an inherited debt level B, there is a level of $\hat{K}$ at which default is triggered. If the firm acquires a new variety, $\hat{K}$ does not change and, so, the probability of default declines. In addition, conditional on default (that is, ending with a $K' \leq \hat{K}$), the expected number of varieties is higher if the firm acquires a new variety, and this increases the recovery on the defaulted debt. On both counts, some portion of the cash flow of the new variety is captured by the firm's existing creditors, thereby blunting the firm's desire to acquire a new variety.

Figure 3:
Increase in Probability of Acceptance without Debt



5 The Risk-Free Rate and Firm Dynamics

In this section, we examine the implications of a lower real interest rate on the steady state equilibrium of our economy. The motivation for this investigation is the well-known decline in the real interest rates and startup rates over the past several decades. Figure 4 shows the secular movement in both the startup rate and the real interest rate over the period 1978-2015.[20] Importantly, as shown in Figure 5, the share of corporate profits in GDP has risen strongly since the late 1990s.[21]

The fact that the decline in startup rates continued, even accelerated, in the face of rising profits is puzzling. But in our model, there is a new margin that effects the entry rate, namely, the choice of organization within which a new idea is implemented. As explained in Section 3, in our economy a decline in r leads to more ideas being implemented in existing firms instead of startups, leading to lower entry rates for firms. Interestingly, Akcigit and Ates (2019, Figure 9, p. 45) document that the share of new patent applications (in total applications each year) that are registered to the top 1 percent of patent holders has risen by around 15 percentage points since the early 1980s, with the bulk of the rise occurring since the mid 1990s. At the same time, the share of new patent applications registered to first-time patenters has declined since the early 1980s with the bulk of the decline again occurring since the mid 1990s. At a broad level, these patenting patterns are consistent with our model implications that a lower interest rate results in the implementation of new ideas moving out of startups and into large firms.

It is to investigate this possibility quantitatively that we chose parameter values to reproduce entry and survival rate statistics from the late 1990s and will now examine how the steady state of our economy changes if the risk-free interest rate declines.[22]

The top panel of Figure 6 shows the trends in the annual short-term real interest rate and the entry rate between 1997 and 2015. The trend line for the real interest rate shows a decline from 2.16 percent (our calibration of r) to -2.16. To understand the effect of lower interest rates, we will

[20] The real TBill rate is the annualized nominal interest rate on 3-month Treasury bills at the start of a year minus the CPI inflation over the previous year. The entry rate is available since 1978 and is from the U.S. Census Bureau's Business Dynamics Statistics database.

[21] The share of corporate profits is the ratio of annual Corporate Profits after Tax (without IVA and CCAdj) to annual Gross Domestic Product published by the U.S. Bureau of Economic Analysis.

[22] Karahan, Pugsley, and Şahin (2019) argue that 50-70 percent of the decline in entry rates between 1978 and 2005 is due to the slowdown in the growth rate of the labor force. We seek to explain the full decline in trend startup rate between 1998 and 2015, which can be viewed as roughly equivalent to explaining the portion of the decline not explained by slower labor force growth.

Figure 4:
Real Interest Rates and Entry Rates, 1978-2015

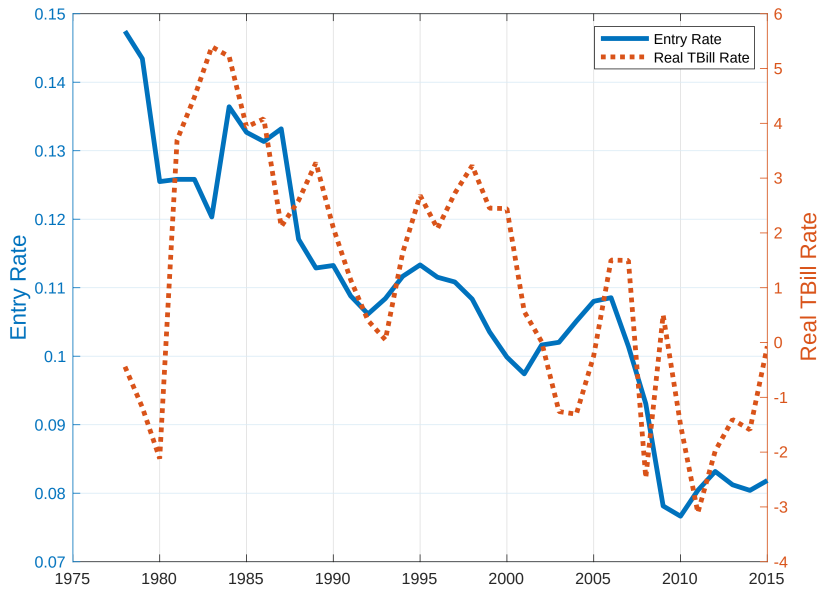


Figure 5:
Entry Rates and Share of Corporate Profits, 1978-2015

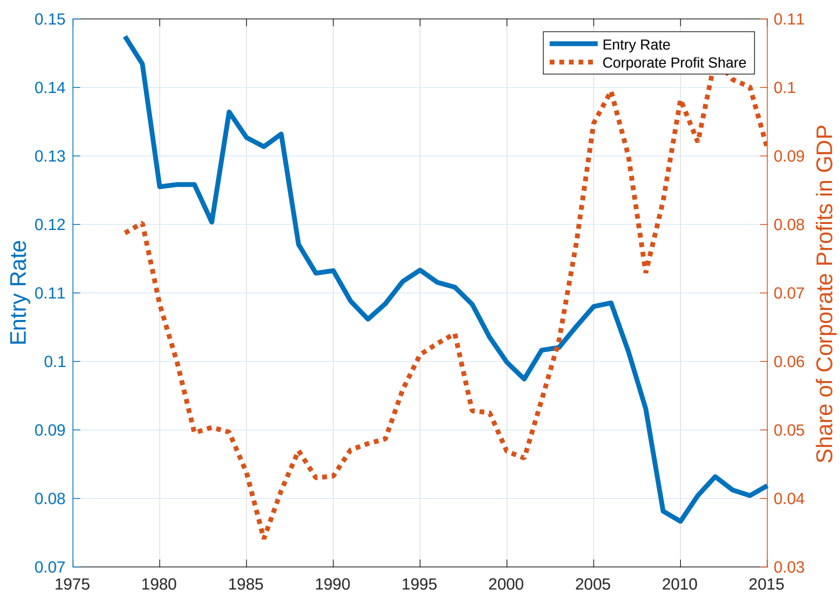
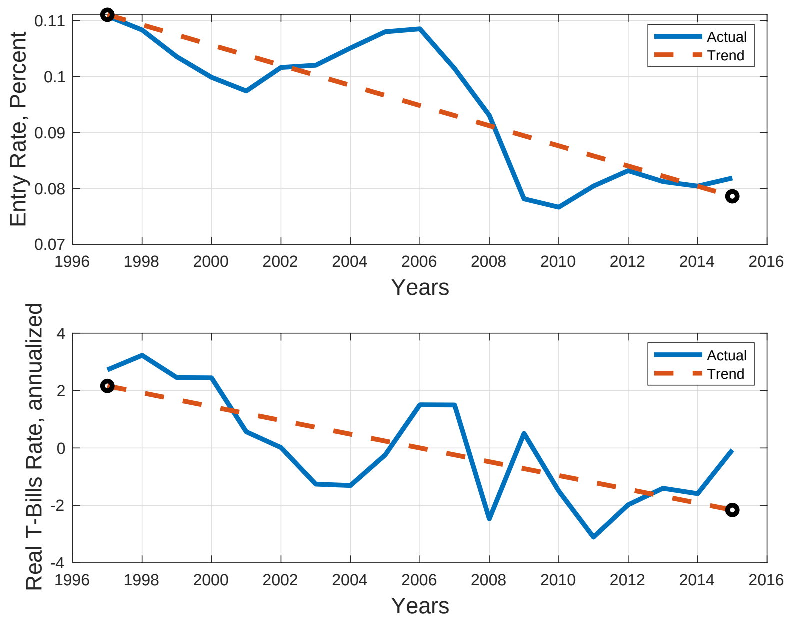


Figure 6:
Trends in Real Interest Rates and Entry Rates, 1997-2015

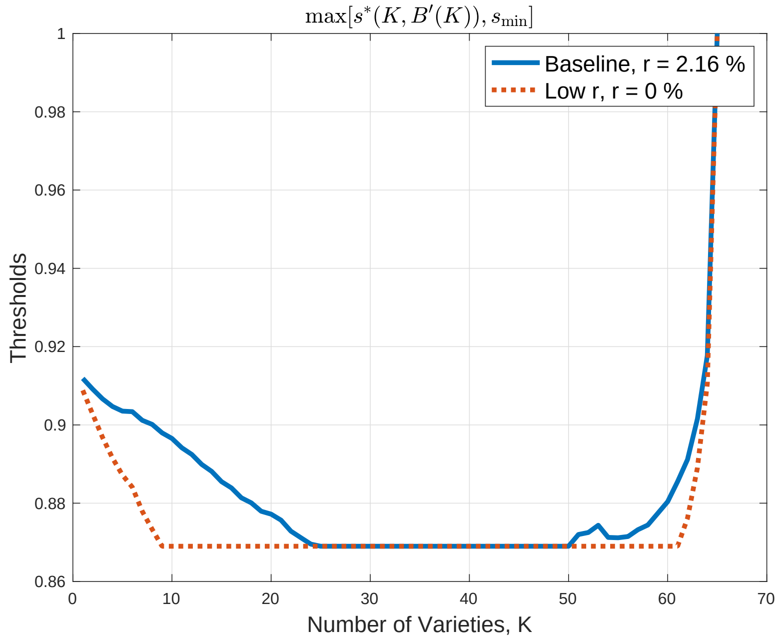


lower interest in our model to 0 percent – keeping all other parameters unchanged — and analyze the new steady state.

When we reduce r from 0.0216 to 0, the startup rate falls from around 0.11 to 0.08. This change is the same as the decline in the trend value of the entry rate between 1997 and 2015, as shown in the bottom panel of Figure 6. This is not a coincidence, as we can control the decline in the entry rate in response to a decline in r by choosing the value of s_{min}. This parameter determines the support of the uniform distribution from which the success probability of an idea for the firm is drawn. If s_{min} is lowered, the range expands, and, therefore, any given decline in the threshold $s^*(\cdot)$ induced by a drop in r has less of an effect on entry rate. Recall that there was a whole set of (σ, s_{min}) values that could generate the startup rate of 0.11, and we set s_{min} to 0.87 and chose σ. The chosen value of s_{min} generates enough sensitivity to explain the declining trend in startups since 1997 as a response to lower real interest rates.

To confirm this effect, Figure 7 plots the threshold value of s above where a firm of size K will purchase an idea. The solid blue line shows this threshold for the baseline model, and the orange dotted line shows it for the equilibrium with $r = 0$ (and no other changes in any parameters). Observe that for each K, the threshold s is either unchanged or lower in the low interest rate

Figure 7:
s Thresholds and the Real Interest Rate



equilibrium. The figure also makes clear that the decline in the risk-free rate is associated with the increase in the size of the region that is associated with misallocation. In the new steady state with lower r, more ideas are implemented in firms with success probabilities lower than startups, leading to a loss in aggregate output.[23]

The decline in interest rates also has the potential to affect business concentration because the decline in the entry rate is accompanied by faster growth of existing businesses. In an influential study, Autor, Dorn, Katz, Patterson, and van Reenen (2017) show that the share of sales in the top 4 or top 20 firms in six major industries has risen since the mid-to-late 1990s. Since our model has a distribution of firms, we can examine the share of sales accounted for by the top *measure* (as opposed to number) of firms. For the baseline model, we use $\mu(K, B)$ to first determine the measure of firms for each K. Then, starting with the firms with the largest number of varieties, we include firms with progressively fewer varieties until 0.1, 0.5, and 1 percent of the total measure of firms is included.

[23]In similar veins, Gopinath, Kalemli-Ozcan, Karabarbounis, and Villegas-Sanchez (2017) argue that, in a world with financial frictions, the decline in interest rates led to an increase in the misallocation of capital and lower productivity in Southern Europe, and Caggese and Perez-Orive (2019) argue that lower interest rates make it harder for firms to accumulate assets necessary to purchase intangible capital that cannot easily serve as collateral for a loan.

Table 3:
Effect on Business Concentration of Low Interest Rates

		Share of Output	
	Measure of firms	Baseline	Low r Eqbm
Top 0.1 percent by Size (K) in Baseline	0.22	0.02	0.03
Top 0.5 percent by Size (K) in Baseline	1.12	0.09	0.13
Top 1.0 percent by Size (K) in Baseline	2.24	0.13	0.24

The first column of numbers in Table 3 reports the resulting measures of firms. Then, we compute the fraction of aggregate cash flow (our measure of output) accounted for by each of the three measures of firms. These fractions are reported in the second column of numbers. Thus, as shown, the top 0.1 percent of firms by size account for 2 percent of total output, the top 0.5 for 9 percent of output, and the top 1 percent for 13 percent of output. For the final column of numbers, using the new distribution of firms for the low interest rate equilibrium, we determine the share of output accounted for by the largest firms for the *measures reported in column 2.* Thus, the comparison between the last two columns holds fixed the number (more precisely, the measure) of top firms. The comparison reveals that the low interest rate economy is substantially more concentrated: For the top (by size) 0.22, 1.12, and 2.24 measures of firms, the share of output rises by 1, 4, and 11 percentage points, respectively.

Table 4 reports some of the other equilibrium effects of a drop in the real interest rate. There is a modest increase in the responsiveness of leverage to sales. One reason underlying this effect is the change in the distribution of firms, which shifts toward larger firms. From Figure 1 presented earlier, it should be reasonably clear that a linear regression of leverage on log of firm size would predict a negative value of leverage for small firms. Indeed, in COMPUSTAT, many small firms have positive net assets because they hold substantial amounts of cash (Opler, Pinkowitz, Stulz, and Williamson (1999) and Duchin (2010)). Our model does not have a reason for savings by firms, and when there are fewer small firms, as in the low interest rate steady state, the relationship between leverage and log sales becomes stronger.

There is a modest increase in the bankruptcy rate in the low interest rate economy, which is also the result of the shift in the distribution toward larger firms. Generally speaking, a small firm's default probability is more sensitive to leverage (i.e., there are bigger upward jumps in probability of default as leverage increases) and, hence, smaller firms are typically further away from θ (the maximum allowed probability of default) in terms of their equilibrium default probability.

The probability that a variety generates a new idea, ρ, increases slightly. This is because existing firms become less choosy about the new ideas they purchase (the s threshold falls) and, consequently, the fraction of new ideas that succeed declines. This leads to a decline in the steady-state measure of varieties. Since the measure of *new* varieties arriving into the economy is constant, this decline translates into an increase in ρ.

Table 4:
Equilibrium Effects of Low Interest Rates

Statistics	Baseline	Low r Eqbm
Response of leverage to sales	0.025	0.037
Fraction of firms that declare bankruptcy	0.012	0.016
Steady-state measure of varieties	516.0	513.0
Prob. of a variety generating a new idea (ρ), ann.	0.209	0.210

Finally, Table 5 reports the model's implications for survival rates for firms of different ages and their employment growth rates. In the model, with lower r, survival rates and employment growth for all age groups increase slightly. The reason for this is because s^* declines and more ideas are implemented within existing firms, increasing their employment growth. Survival rates also go up because existing firms are less likely to lose all their varieties and exit. But overall, the changes are quite small. In the data, we see that survival rates and employment growth do increase for firms that are one year old or older, and the change is more pronounced than what the model generates. For 0-year-old firms, both survival rates and employment growth shrink. This might be because, in a world where there is competition between existing firms for new ideas, young firms may become more disadvantaged relative to large firms as interest rates decline. Our model does not take such effects into account.

Table 5:
Survival Rates and Employment Growth By Firm Size

Statistics		Baseline	Low r Eqbm	
	Data 1997	Baseline	Low r Eqbm	Data 2015
Survival rate of 0-yr-old firms	0.77	0.83	0.83	0.76
Survival rate of 1-yr-old firms	0.84	0.84	0.85	0.87
Survival rate of 2-yr-old firms	0.87	0.86	0.86	0.89
Survival rate of 3-yr-old firms	0.88	0.87	0.87	0.91
Survival rate of 4-yr-old firms	0.90	0.88	0.88	0.91
Employment growth of 0-yr-old firms	0.99	0.94	0.94	0.90
Employment growth of 1-yr-old firms	0.92	0.94	0.94	0.95
Employment growth of 2-yr-old firms	0.93	0.94	0.95	0.97
Employment growth of 3-yr-old firms	0.94	0.94	0.95	0.98
Employment growth of 4-yr-old firms	0.96	0.94	0.95	0.96

The data for survival rates and employment growth are from the U.S. Census Bureau's Business Dynamics Statistics database (https://www.census.gov/ces/dataproducts/bds/data.html). Each data point reported is the predicted trend from a linear time trend regression between 1997 and 2015 of the corresponding series.

6 A Default Cost Model

The goal of this section is to show that the constraint on default probabilities is not necessary for the main results. We study an enviroment in which lenders incur a fixed cost $\Delta > 0$ in the event of default. In this model, even if θ is set equal to 1, there is a difference in the access to external finance for small and large firms.

All equations are as in the main text, except that the equilibrium condition for the price of debt is now

$$
\begin{aligned}
&q(K', B')(1+r) \qquad (11) \\
&= \left[\rho K' \int s \mathbb{1}_{\{s \geq s^*(K', B')\}} dF(s)\right] \mathbb{E}_{(K''|K'+1)} \left[[1 - D(K'', B')] + D(K'', B') \frac{\bar{B}(K'') - \Delta}{B'}\right] + \\
&\quad \left[1 - \rho K' \int s \mathbb{1}_{\{s \geq s^*(K', B')\}} dF(s)\right] \mathbb{E}_{(K''|K')} \left[[1 - D(K'', B')] + D(K'', B') \frac{\bar{B}(K'') - \Delta}{B'}\right],
\end{aligned}
$$

where, as before, $\bar{B}(K'') = \pi K'' + G(K'')$.

Table 6

Statistic	Base Model	Default Cost Model
Probability of default	0.01	0.0001
Annual entry rate of new firms	0.11	0.11
Survival rate of 1-yr-old firms	0.84	0.84
Equilibrium value of ρ	0.209	0.210

In a competitive world, creditors have to be compensated for the loss of Δ in the event of default, which means that firms must pay a higher interest rate in the state of the world in which they do not default. Since default is more likely for smaller firms for any level of debt, they must pay a higher interest rate (obtain a lower price) than larger firms for the same level of debt.[24]

Remarkably, this model is as capable as the model in the main text in accounting for the leverage-size relationship: Other model statistics barely change, except that the average default probability is much lower. The fixed cost model has difficulty accounting simultaneously for both the default rate and the response of leverage to firm size, which is a consequence of the low leverage of small firms relative to the base model. However, the effect of a drop in the risk-free rate (from 2.16 to 0) is almost identical to that of the main model: The entry rate of firms declines from 0.11 to 0.08.

7 Conclusion

We presented a model in which firms manage collections of product varieties. The arrival into the economy of new varieties and the extinction of existing varieties are random events. Since firms manage collections of varieties, the random process of *product variety* entry and exit induces a stochastic process for the entry, growth, and exit of *firms*. A firm's access to capital markets plays a key role in our theory of firm dynamics. Our model generates a positive relationship between firm size and firm leverage that is consistent with the evidence for U.S. firms. Our theory implies that a decline in the risk-free rate will result in larger firms purchasing more of the new varieties entering the economy in any period, resulting in fewer startups and greater concentration of sales among top firms. Thus, our paper connects the decline in the startup rate and the rise in business concentration since the late 1990s to the decline in the risk-free rate over this same period.

[24]Furthermore, for the same default risk, the fixed cost makes the interest rate on smaller loans higher.

Appendix A

This Appendix describes the choice problem of a firm with $(N + 1) > \rho K \geq N$. Such a firm gets the opportunity to buy N ideas for sure and the opportunity to buy the $(N + 1)$st idea with probability $\rho K - N$. At each purchase node, the firm knows whether its previous purchases (if any) were successful or not. The case $N = 0$ was covered in the text. Here we generalize to any $N \geq 0$.

Given $K \in \mathbb{K}$, let $N(K)$ satisfy $N(K) + 1 \geq \rho K > N(K)$.

Let $j \in \{1, 2, \ldots, N(K)\}$ be the order of the purchase node.

At $j = 1$ (the first purchase node) the number of varieties owned by the firm is K, the number owned at the end of the previous period. At purchase node $j > 1$, K_j can be any number in $\{K, K + 1, \ldots, K + (j - 1)\}$, depending on how many of the past purchases in the current period have been successful.

- For $j = N(K)$ and $K_j \in \{K, K + 1, \ldots N(K) - 1\}$, let

$$Z_j(K_j, B; N(K))) = [N(K) - \rho K] W(K_j, B) \ + [\rho K - N(K) - 1] \times \int_{s_{\min}}^{1} [\max\{W(K_j, B), sW(K_j + 1, B) + (1 - s)W(K_j, B) - \sigma W(1, 0)\}] \, dF(s).$$

  Here $W(N, B)$ has the same interpretation as in the main text: It is the value of the firm after the merger decisions have been made but before the product extinction shocks are realized.

- For $j \in \{1, \ldots, N(K) - 1\}$ and $K_j \in \{K, K + 1, \ldots, K + j - 1\}$, let

$$Z_j(K_j, B; N(K)) = \int_{s_{\min}}^{1} [\max\{Z_{j+1}(K_j, B; N(K)), sZ_{j+1}(K_j + 1, B; N(K)) + (1 - s)Z_{j+1}(K_j, B; N(K)) - \sigma W(1, 0)\}] \, dF(s).$$

- Finally, let

$$Z(K, B) = Z_1(K, B; N(K)).$$

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