WORKING PAPERS
RESEARCH DEPARTMENT

WORKING PAPER NO. 16-31
REGULATING A MODEL

Yaron Leitner
Research Department
Federal Reserve Bank of Philadelphia

Bilge Yilmaz
The Wharton School
of the University of Pennsylvania

October 2016

RESEARCH DEPARTMENT, FEDERAL RESERVE BANK OF PHILADELPHIA
Ten Independence Mall, Philadelphia, PA 19106-1574 • www.philadelphiafed.org/research-and-data/

Regulating a Model*

Yaron Leitner[†] Bilge Yilmaz[‡]

October 26, 2016

Abstract

We study a situation in which a regulator relies on models produced by banks in order to regulate them. A bank can generate more than one model and choose which models to reveal to the regulator. The regulator can find out the other models by monitoring the bank, but, in equilibrium, monitoring induces the bank to produce less information. We show that a high level of monitoring is desirable when the bank's private gain from producing more information is either sufficiently high or sufficiently low (e.g., when the bank has a very little or very large amount of debt). When public models are more precise, banks produce more information, but the regulator may end up monitoring more.

Keywords: bank regulation, Bayesian persuasion, internal-risk models, model-based regulation

JEL Classifications: D82, D83, G21, G28

*The views expressed in this paper are those of the authors and do not necessarily reflect those of the Federal Reserve Bank of Philadelphia or of the Federal Reserve System. For helpful comments, we thank Gilles Chemla and Leonid Spesivtsev, as well as seminar participants at Wharton, the Federal Reserve Bank of Philadelphia, the Barcelona GSE summer forum workshop on financial intermediation and risk, the Brevan Howard/FTG conference, and the Asian meetings of the Econometric Society. This paper is available free of charge at www.philadelphiafed.org/research-and-data/publications/working-papers/.

[†]Research Department, Federal Reserve Bank of Philadelphia [‡]Finance Department, Wharton School of the University of Pennsylvania

1 Introduction

Regulators rely on information produced by banks in order to regulate them. For example, under the Basel II Accord, banks' internal risk models can be used to calculate regulatory capital. The idea behind this approach is that banks may know more than regulators about their own risk. After all, banks have strong incentives to develop good models for their own trading. Yet, a concern exists that while banks will use their best models for trading purposes, for the purpose of regulation, they will select models that underestimate risk. This concern, which is supported by empirical evidence,[1] has led regulators to rethink whether the risk estimates that banks provide should be used to calculate regulatory capital. Recent proposals call to limit, or even stop, this practice.[2]

The first question we address in this paper relates to the concern above: Should regulators attempt to find out all relevant information from banks, even if regulators can do so without incurring any cost? In our model, a bank can create more than one model and choose which models to reveal to the regulator. The regulator uses the information from the models he observes to decide whether to allow the bank to invest in some risky asset. The regulator also decides how much to monitor the bank, which leads to an endogenous probability q that the regulator will find out the other models that the bank generates.

As we explain below, there are two forces that push the optimal q in different directions. A higher q allows the regulator to learn more from the information that the bank produces. This can lead to better investment decisions from the regulator's point of view. However, a higher q may also induce the bank to produce less information overall (in the sense of Blackwell, 1951). This is because if the regulator finds out the information, he can use it to restrict investment when the bank wants to invest but the regulator does not.

[1] See, for example, Behn, Haselmann, and Vig (2014); and Plosser and Santos (2014).

[2] See, for example, "Regulator Suggests End to Banks'Self-Grading," by Peter Eavis, *New York Times*, May 8, 2014; and "Basel Committee to Stop Banks Gaming Risk Models," by Archie Van Reimsdjik, *Wall Street Journal*, November 2, 2015.

The conflict of interest between the bank and the regulator could arise for various reasons. Banks do not internalize externalities, while regulators are concerned with systemic risk and take into account total social costs. Similarly, banks have limited liability, and, as such, shareholders do not necessarily maximize total firm value. Our main insights do not depend on the specific source of conflict.

For concreteness, we focus on one source of conflict. In our model, the regulator wants to invest only if a project has a positive net present value (NPV), but the bank, facing limited liability, wants to invest also in some projects that have negative NPV. This conflict of interest is also impacted by the fact that the bank has other assets, which could be used to pay debt holders when the project fails. Since both the value of these assets and the value of the project depend on the state of the world, the bank's ideal investment rule may not be just a simple cutoff rule. For example, it may be optimal for a bank to invest in good states (i.e., when the project has positive NPV) and bad states (i.e., when the project has negative NPV and other assets are worthless) but not in intermediate states (i.e., when the value of the other assets is too high to lose). More generally, the bank's ideal investment rule could include intervals of the state space in which the bank wants to invest and intervals in which the bank does not want to invest.

The bank can generate information to guide its investment decisions. Specifically, the bank generates *models*, which are information partitions of the state space. We restrict attention to information partitions, such that each element in the partition is a convex set (i.e., an interval or a singleton). Then, the bank faces a tradeoff. When the bank generates more information, the bank can make better investment decisions for its equity holders. However, if the regulator finds out the information, the regulator can use the information against the bank to ban investment when the bank wants to invest but the regulator does not. The outcome of this tradeoff is that, when the regulator sets a higher level of monitoring, the bank produces less information. Consequently, the optimal level of monitoring could have an interior solution.

We characterize the optimal level of monitoring. In particular, we provide necessary and sufficient conditions under which it is optimal to set $q = 1$. When these conditions hold, it is optimal that the regulator observes all the models that the bank creates. When these conditions do not hold, it is optimal that the bank creates two sets of models. The bank reveals to the regulator the first set of models but does not reveal the second set of models. The regulator allows the bank to invest as long as he is convinced that the state of the world is sufficiently high for the project to have a nonnegative NPV, on expectation. The bank maintains discretion whether to invest when the regulator allows it to invest.

Interestingly, there is a nonmonotone relationship between the optimal level of monitoring and the bank's private gain from producing more information. When the bank's private gain is high, the bank produces a lot of information even if it is highly monitored. In this case, it is optimal to set $q = 1$ because the regulator can learn everything the bank knows without impacting the amount of information that the bank produces. If instead, the bank's gain is intermediate, the regulator must set a lower level of monitoring $q < 1$ to induce the bank to produce more information. In this range, the optimal level of monitoring decreases when the bank's gain from producing information falls because it becomes harder to induce the bank to produce information. Finally, when the bank's private gain from producing information is very low, the regulator can induce the bank to produce information only if the level of monitoring is very low. But then the regulator does not learn much from the information that the bank produces, and it is again optimal to set $q = 1$, even though this reduces the overall amount of information that the bank produces.

Using this insight, we derive comparative statics as to how the optimal q changes when the bank faces some exogenous cost of producing information, when the amount of its debt holding changes, when the quality of its project changes, or when the value of its existing asset changes. In general, the relationship is nonmonotone. For example, for some parameter values, it is optimal to set a high level of monitoring for banks that have either low cost or high cost of producing information, and a lower level of monitoring for banks that have a medium cost. Similarly, banks that have either high levels of debt or low levels of debt could face a high level of monitoring, while banks with a medium level of debt could face a lower level of monitoring. As for the amount of information produced, our model predicts that for a given (positive) level of monitoring, banks will produce less information when they have more debt, when the value of their existing assets falls, and, perhaps surprisingly, when they have higher quality projects. All these changes increase the bank's gain from investing, and, hence, the fact that the regulator could use the information to ban investment has a more significant effect on the bank.

We also analyze a situation in which some public models already exist. We show that when public models become more informative, the bank generates more information. However, there is a nonmonotone relationship between the quality of the public models and the optimal level of monitoring. Intuitively, there are two forces that push the optimal level of monitoring in different directions. On the one hand, with more public information, the regulator can monitor less because he already has information. On the other hand, public information can increase the bank's private gain from producing information because the alternative of not producing information is less attractive, as information already exists. This makes it easier for the regulator to induce the bank to produce information and allows the regulator to increase the probability of monitoring.

The paper proceeds as follows. The next section provides a literature review. Section 3 provides an example, and Section 4 presents the formal model. In Section 5, we analyze the benchmark case of an unregulated bank, and, in Section 6, we provide equilibrium analysis of a regulated bank. In Section 7, we do some comparative statics, and in Section 8, we analyze the case of public models. Section 9 illustrates how the insights of our model can be applied in other settings, such as corporate governance or when regulators use banks' internal risk models to calculate regulatory capital. Section 10 concludes. Proofs are in the Appendix.

2 Literature review

One can think of our paper as a Bayesian persuasion problem (Kamenica and Gentzkow, 2011), in which one agent (the bank) generates a signal (a model) to persuade another agent (the regulator) to allow some action. A key difference between our paper and existing literature is that, in our setting, the bank can generate a second signal, which it does not reveal to the regulator. The bank can then use information from both signals to decide whether to take the action. Our paper allows us to answer questions such as whether and under what conditions the regulator can gain by committing not to find out what the second signal is.

Our paper also relates to the literature on delegation of authority within organizations, in particular, the literature that focuses on the tradeoff between incorporating more information into decision making and controlling decision-making authority when the agent has relevant information. In a strategic communication setting, Dessein (2002), Harris and Raviv (2005, 2008, and 2010), Chakraborty and Yilmaz (2014), and Grenadier, Malenko, and Malenko (2016) study the conditions under which the principal should allocate decision-making authority to the agent. In this framework, keeping decision-making authority may hurt the principal because the gains from doing so are outweighed by the losses arising from imperfect information transmission between the agent and the principal. In a related work, Aghion and Tirole (1997) also analyze the optimal allocation of authority but without strategic communication while emphasizing a distinction between formal and real authority. They show that often the party with formal authority will delegate authority to another agent with information. In our framework, the regulator can gain by delegating authority to the bank. A key difference between our paper and the existing literature is that in our setting, the regulator allocates authority based on the realization of a signal that the bank produces (endogenously). In particular, if the realization of the signal(s) that the bank reveals to the regulator is above some threshold and the regulator does not observe the other signal(s), the regulator allows the bank to decide whether to invest. Otherwise, the regulator decides.

Our paper also relates to the literature on bank regulation in which the regulator uses information that banks provide to set capital requirements. Prescott (2004) studies a framework in which the bank has private information about its own risk, and the regulator sets capital requirements based on the risk that the bank reports.[3] The source of inefficiency is that the bank may misrepresent its true risk. The regulator can mitigate this problem by monitoring the bank.[4] In our framework, the bank does not know its own risk but can generate information. In contrast to earlier literature on the benefits of monitoring, our model shows that too much monitoring could have perverse effects. Monitoring diminishes the banks' incentive to produce valuable information.

There is also a growing empirical literature on the effect of regulation that relies on banks'internal models. Plosser and Santos (2014) found systematic differences in risk estimates that large U.S. banks provided for the same loan. In particular, banks with lower regulatory capital reported lower probability of default, and the prices they set on their loans were poorly explained by their risk estimates. Behn, Haselmann, and Vig (2014) compare risk estimates that German banks provided on loan portfolios that shifted to the internal rating-based approach and loan portfolios that were waiting for approval for the new approach and for which capital was calculated based on the traditional risk-weights method. They show that, for the first group of loans, banks provided lower estimates of probability of default. Yet, the interest rates charged on these loans and actual default rates were higher. The findings of this literature are consistent with the idea that banks hide information from the regulator.[5]

Finally, the idea that a principal can benefit by committing not to monitor too much also appears in Crémer (1995). In his model, a principal has a monitoring technology, which allows him to obtain the reason behind a low output. The prin-

[3] See also Marshall and Prescott (2006).

[4] Blum (2008) shows that when the regulator has limited ability to audit or impose penalties, minimum leverage ratios in addition to capital requirements can help.

[5] Some other related work include Begley, Purnanandam, and Zheng (2016), Firestone and Rezende (2016), and Mariathasan and Merrouche (2014).

cipal may benefit from inefficient monitoring because it allows him to pre-commit to fire a high-quality agent who produced low output.[6] In the context of regulation of disclosure in the product market, Polinsky and Shavell (2012) show that forcing firms to disclose information about product risk may lead firms to gather less information.[7] In the context of corporate governance, Burkart, Gromb, and Panunzi (1997) show that excessive monitoring can reduce managerial effort to learn about new investments, and that dispersed ownership could act as a commitment device not to exercise excessive control.

3 An example

There is a bank and a regulator. The bank has a debt liability with a face value of \$1. The bank also has \$1 in cash, other existing assets, and a new investment opportunity (project). The project requires an investment of \$1. It can either succeed and yield \$2 or fail and yield nothing. The project's success probability and the value of the bank's existing assets (other than cash) depend on the unobservable state of nature, as described in Table 1. Table 1 also shows the project's NPV in each state, namely, the project's expected payoff minus the initial investment.

Table 1

State	s_1	s_2	s_3	s_4
Probability of state	0.25	0.25	0.25	0.25
Project's success probability	0.1	0.4	0.4	0.8
Value of existing assets	0.3	0.3	0.8	1
Project's NPV	-0.8	-0.2	-0.2	0.6

There is a conflict of interest between the bank and the regulator. The regulator wants to maximize total surplus, which is the sum of payoffs to debt holders and equity holders. Hence, the regulator wants to invest only if the project has positive NPV. The bank has limited liability and acts as to maximize the expected payoff to its equity holders. Hence, as we illustrate below, the bank wants to invest not only

[6]See also Cohn, Rajan, and Strobl (2013), who show that when credit rating agencies screen more heavily, issuers have stronger incentive to manipulate.

[7]See also Shavell (1994).

when the project has positive NPV but also in some states in which the project has negative NPV.

To see that, consider state s_2. If the bank does not invest, debt holders are fully paid. However, in case of investment, if the project fails, which occurs with probability 0.6, the bank cannot pay off its debt. In this case, the bank's debt holders obtain the bank's existing assets, which are worth only 0.3. The expected loss for debt holders due to investment in state s_2 is then $0.6 \times (1-0.3) = 0.42$. From the perspective of the bank's equity holders, this is beneficial because this is a transfer of wealth from debt holders. Since the sum of this gain (0.42) and the project's NPV in state s_2 (-0.2) is positive, the bank wants to invest in state s_2. Table 2 repeats these calculations for the other three states. It follows that the bank wants to invest in states s_2 and s_4 but not in states s_1 and s_3.

Table 2

State	s_1	s_2	s_3	s_4
Gain from default	0.63	0.42	0.12	0
Project's NPV + Gain from default	-0.17	0.22	-0.08	0.6
Bank's ideal investment rule	Don't invest	Invest	Don't invest	Invest

If the regulator knew the state, he would allow the bank to invest only in state s_4. However, the regulator is not an expert in producing information. The only way for the regulator to learn more about the state is to rely on information that the bank produces.

The bank can generate two types of signals. The first signal is very informative. It fully reveals the state. The second signal is less informative. It tells only whether the success probability is 0.1 or above 0.1. In other words, the second signal tells whether the true state is in the set $\{s_1\}$ or in the set $\{s_2, s_3, s_4\}$. The regulator cannot dictate to the bank which signal (or signals) to generate, but he can force the bank to disclose the signal realization. Putting it differently, the regulator has a monitoring technology that allows him to fi nd out the signals that the bank generates. Note that if the bank does not generate any signal, the regulator would ban investment because the project' sexpected NPV, averaged across all states, is negative.

Suppose first that the regulator monitors the bank. Which signal will the bank generate? If the bank generates the very informative signal, the regulator would allow it to invest only in state s_4. The expected gain for the bank's equity holders is then 0.25×0.6. If the bank generates the less informative signal, the regulator will ban investment when he learns that the state is s_1 but will allow investment when the state is in $\{s_2, s_3, s_4\}$. The last part follows because conditional on being in $\{s_2, s_3, s_4\}$, the project has a positive NPV. The bank will then invest in the three states s_2, s_3, s_4, yielding an expected gain of $0.25 \times (0.22 - 0.08 + 0.6)$ to its equity holders. Hence, the bank will generate the less informative signal, as it leads to a higher gain for its equity holders.

Now suppose the bank can generate both signals and reveal to the regulator only the less informative one. In other words, the regulator does not monitor the bank. As before, the regulator will allow investment only in states $\{s_2, s_3, s_4\}$. However, now the bank can use the information from the more informative signal to decide whether to invest. So, the bank will invest only in states s_2 and s_4, in which the gain to its equity holders is positive. From the regulator's point of view (and also from the bank's), this outcome is preferred to the outcome when the bank is monitored because the bank does not invest in state s_3. Hence, the regulator will not monitor the bank.

The result above changes when the bank has less debt. Suppose the face value of debt is only \$0.8. Now the bank's equity holders gain from defaulting on the bank's debt only in states s_1 and s_2, in which the value of existing assets is less than the face value of debt. As we show in Table 3, the bank will still want to invest only in states s_2 and s_4. However, the bank's incentives to produce information change. Now the bank will produce the more informative signal even if it is being monitored. To see why, note that if the bank generates the more informative signal, it obtains 0.25×0.6, as in the case of debt with a face value of 1. If the bank generates the less informative signal, it obtains only $0.25 \times (0.1 - 0.2 + 0.6)$. Hence, the bank will generate the more informative signal, although it knows that it will be forced to reveal the information. From the regulator's perspective, this is the best possible outcome.

Hence, the regulator will monitor the bank.

Table 3

State	s_1	s_2	s_3	s_4
Gain from default	0.45	0.3	0	0
Project's NPV + Gain from default	-0.35	0.1	-0.2	0.6
Bank's ideal investment rule	Don't invest	Invest	Don't invest	Invest

We can interpret the signals in this example as internal risk models that the bank generates for regulatory purpose. The example illustrates three points. First, the regulator could benefit from relying on internal risk models that the bank generates. Second, the regulator could gain by allowing the bank to produce two sets of models. The first model is used to persuade the regulator to allow the bank to invest, while the second model — which is not shared with the regulator — is used by the bank to decide whether to invest when the regulator allows it to do so. Third, whether the regulator could gain by allowing the bank to produce two sets of models depends on the bank's private gain from producing information, which, in turn, depends on bank characteristics, such as how much debt the bank has.

4 The model

The formal model generalizes the example in Section 3. In particular, we assume a continuous state space and allow the bank to choose from a larger set of signals. We refer to these signals as models. In addition, we allow for partial monitoring, which induces a probability $q \in [0, 1]$ that the regulator will find out all the models that the bank has generated.

4.1 Economic environment

As in the example, the bank's assets consist of cash, which is normalized to 1, a risky asset, and a new investment opportunity (project). The value of the risky asset depends on the unobservable state ω, according to some continuous function $v(\omega)$. The state ω is drawn from the set $\Omega = [0, 1]$, according to a continuous cumulative distribution function F (everything is common knowledge). The value of the new project also depends on the state. The new project requires an investment of 1. It generates $x > 1$ with probability ω and 0 with probability $1 - \omega$.[8] The bank also has a debt liability with face value $D \leq 1$. The bank has limited liability.

The bank acts to maximize the expected payoff to its equity holders. The regulator maximizes total surplus, which is the sum of payoffs to debt holders and equity holders.

4.2 Information production

The bank can generate information about ω by creating *models*. A model is an information partition of Ω, with the added requirement that each element in the partition is a convex set. Formally:

Definition 1 *A model is defined by a set of indexes $\mathcal{I} \subset R$ and a collection of sets $\mathbf{P} = \{P_i\}_{i \in \mathcal{I}}$, such that the following hold:*

1. $\cup_{i \in \mathcal{I}} P_i = \Omega$.

2. For every $i \neq j$, $P_i \cap P_j = \emptyset$.

3. For every $P_i \in \mathbf{P}$ and $\lambda \in (0, 1)$, if $\omega_1, \omega_2 \in P_i$, then $\lambda\omega_1 + (1 - \lambda)\omega_2 \in P_i$.

The convexity assumption (Part 3 in Definition 1) implies that each element is either a singleton or an interval. We let $P(\omega)$ stand for the set in $\mathbf{P}$ to which ω belongs. So, when the realized state is $\omega \in \Omega$, the model $\mathbf{P}$ tells that the event $P(\omega)$ has occurred. For example, a model that consists of only singletons fully reveals ω. A model that consists of two intervals tells whether ω is above or below some threshold. We can think of a model as a collection of experiments, where each experiment tells whether the state is above or below some threshold.

The bank can create more than one model. With probability q, the regulator observes all the models that the bank creates. With probability $1 - q$, the regulator observes only the models that the bank chooses to reveal. The probability q is

[8] The nature of the results remains if the project's probability of success depends on the state according to some arbitrary function that is (weakly) increasing in the state.

endogenous and determined by the regulator before the bank creates models. For example, a higher q could capture the idea that the regulator devotes more resources into monitoring the bank (e.g., by having more staff on site). In this case, it is natural to assume that the regulator can precommit to acting according to q. For simplicity, we assume that all choices of q entail the same cost.[9] The regulator can allow or ban investment based on the information it has about ω. However, the regulator cannot precommit to investment rules that are suboptimal ex post.

Without loss of generality, we can assume that the bank generates only two models $\mathbf{P}^B$, $\mathbf{P}^R$ such that model $\mathbf{P}^B$ contains all the information that the bank produces, and model $\mathbf{P}^R$ contains only the information that the bank chooses to reveal to the regulator. In particular, if the bank creates m models $\mathbf{P}^1, ..., \mathbf{P}^m$ and reveals to the regulator only the first $l \leq m$ models, we can define for every $\omega \in \Omega$, $P^B(\omega) = \cap_{j=1}^m P^j(\omega)$ and $P^R(\omega) = \cap_{j=1}^l P^j(\omega)$. We refer to models $\mathbf{P}^B$ and $\mathbf{P}^R$ as the bank model and regulator model, respectively. Note that $\mathbf{P}^B$ is at least as informative as $\mathbf{P}^R$.

4.3 Sequence of events

The sequence of events is as follows:

1. The regulator chooses $q \in [0, 1]$ and publicly announces it.
2. The bank chooses models $\mathbf{P}^R$ and $\mathbf{P}^B$.
3. Nature draws the state ω. The bank observes $P^B(\omega)$. With probability q, the regulator observes $P^B(\omega)$. With probability $1-q$, the regulator observes $P^R(\omega)$.
4. The regulator allows or bans investment.
5. If investment is allowed, the bank chooses whether to invest.
6. The project either succeeds or fails. Debt holders and equity holders get paid.

[9] This assumption helps us focus on the main tradeoff in our paper. It is easy to relax this assumption, but relaxing this assumption does not provide any interesting insights.

We focus on perfect Bayesian equilibria of the game above. Assume that if the bank is indifferent between investing and not investing, the bank invests. If the regulator is indifferent between allowing and banning investment, the regulator allows investment.

5 Unregulated bank (benchmark)

We start with the benchmark case in which the bank is unregulated; that is, the regulator cannot ban investment. In this case, we can assume, without loss of generality, that the bank generates only one model, which fully reveals the state. That is, for every $\omega \in \Omega$, $P^R(\omega) = P^B(\omega) = \omega$.

We derive the bank's ideal investment rule as follows. If the bank does not invest, debt is riskless, and the bank's equity holders obtain

$$v(\omega) + 1 - D. \tag{1}$$

If the bank invests, debt holders obtain D when the project succeeds and $\min\{v(\omega), D\}$ when the project fails. So, if the bank invests, the expected payoff to the bank's equity holders is

$$\omega[x + v(\omega) - D] + (1 - \omega)\max\{v(\omega) - D, 0\}. \tag{2}$$

Denote the project's NPV in state ω by

$$N(\omega) \equiv \omega x - 1.$$

The expected gain to the bank's equity holders from investing in state ω [i.e., (2) minus (1)] is:

$$G(\omega) \equiv N(\omega) + (1 - \omega)\max\{D - v(\omega), 0\}. \tag{3}$$

The second term in (3) is the expected gain to the bank's equity holders from defaulting on the bank's debt when the project fails. This gain arises when the value of the risky asset is less than the face value of debt. In this case, equity holders benefit at the expense of debt holders, who get paid less than the promised amount.

The bank invests if and only if $G(\omega) \geq 0$.

Lemma 1 *$G(\omega) \geq 0$ if and only if either (i) $\omega \geq \frac{1}{x}$; or (ii) $\omega < \frac{1}{x}$ and $v(\omega) \leq D + \frac{N(\omega)}{1-\omega}$.*

Lemma 1 says that an unregulated bank invests if either (i) the project has positive NPV; or (ii) the project has negative NPV, but the value of the bank's existing asset, $v(\omega)$, is less than $D + \frac{N(\omega)}{1-\omega}$.

Figure 1 illustrates the function $D+\frac{N(\omega)}{1-\omega}$, which is convex and increasing in ω, and the function $v(\omega)$. The bank's ideal investment rule depends on how the two functions intersect. In general, the bank's ideal investment rule is composed of intervals in which the banks invests and intervals in which the bank does not invest. To simplify the exposition, we assume that the functions $D + \frac{N(\omega)}{1-\omega}$ and $v(\omega)$ intersect at a finite number of points. Then, there exist a finite set of numbers $b_1 > a_1 > ... > b_l > a_l$, such that $G(\omega) \geq 0$ if and only if $\omega \in \cup_{i=1}^{l}[a_i, b_i]$.

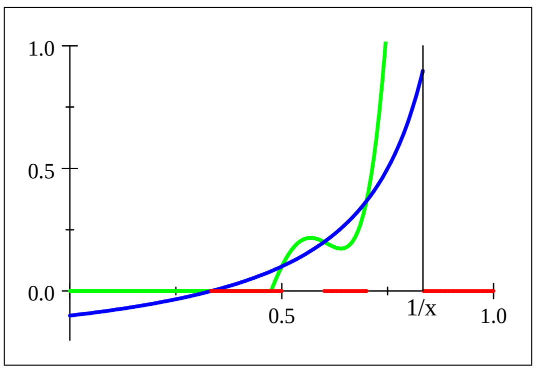

Figure 1. The figure illustrates the function $v(\omega)$ (in light green) and the function $D + \frac{N(\omega)}{1-\omega}$ (in blue). The bank's ideal investment rule is to invest when the state ω is in the red intervals. This happens when either (i) $\omega \geq \frac{1}{x}$, so the project has positive NPV; or (ii) $\omega < \frac{1}{x}$ and the green line is below the blue line.

6 Regulated bank

Consider now a regulated bank. If $v(\omega) > D + \frac{N(\omega)}{1-\omega}$ for every $\omega < \frac{1}{x}$, it follows from Lemma 1 that the bank wants to invest if and only if the project has positive NPV. In this case, there is no conflict of interest between the bank and regulator, and so, regulation is unnecessary.

The rest of this paper focuses on the case in which $v(\omega) \le D + \frac{N(\omega)}{1-\omega}$ for some $\omega < \frac{1}{x}$. In this case, the bank and the regulator do not agree on the investment rule. Both want to invest when the project has a positive NPV ($\omega \ge \frac{1}{x}$), but the bank wants to invest also in some states in which the project has a negative NPV [Part (ii) in Lemma 1]. We explore optimal regulation in this case. We first characterize equilibrium outcomes, taking q as given. Then, we solve for an optimal q (i.e., a q that the regulator chooses in an equilibrium).

6.1 Equilibrium outcomes for a given q

We solve the game backward. Suppose the bank chooses models $\mathbf{P}^R, \mathbf{P}^B$. If the regulator allows investment, the bank invests if and only if the expected gain to its equity holders is positive. Anticipating the bank' s behavior, the regulator allows investment if and only if he expects the project to have a positive NPV when the bank invests.

The next lemma simplifies the analysis.

Lemma 2 *For any equilibrium outcome, there exists $\omega_B, \omega_R \in \Omega$, such that $\omega_R \le \omega_B$ and:*

(i) When the regulator observes model $\mathbf{P}^B$, investment takes place if $\omega > \omega_B$ but not if $\omega < \omega_B$.

(ii) When the regulator observes model $\mathbf{P}^R$, (a) investment takes place if $\omega > \omega_B$; (b) investment does not take place if $\omega < \omega_R$; and (c) if $\omega \in (\omega_R, \omega_B)$, investment takes place when $G(\omega) > 0$ but not when $G(\omega) < 0$.

The first part in Lemma 2 follows because the project's NPV is increasing in ω, and each set in the model partition is convex. This implies that if the regulator allows the bank to invest in state ω', he also allows the bank to invest in higher states $\omega > \omega'$. Moreover, since the bank and the regulator share the same information, the bank invests if the regulator allows it to because the regulator allows investment only if the project has nonnegative NPV, in expectation. If, instead, the regulator observes only $\mathbf{P}^R$, as in the second part in Lemma 2, the regulator allows investment when $\omega \geq \omega_R$, but now the bank decides whether to invest based on information from the more informative model $\mathbf{P}^B$. In equilibrium, the bank chooses model $\mathbf{P}^B$, so that whenever $\omega \in (\omega_R, \omega_B)$, the bank invests according to its ideal investment rule: namely, if $G(\omega) > 0$ but not if $G(\omega) < 0$.

Lemma 2 implies that, without loss of generality, we can focus on the case in which models $\mathbf{P}^B$ and $\mathbf{P}^R$ take the simple form:

$$P^B(\omega) = \begin{cases} \omega & \text{if } \omega < \omega_B \\ [\omega_B, 1] & \text{otherwise.} \end{cases} \tag{4}$$

$$P^R(\omega) = \begin{cases} \omega & \text{if } \omega < \omega_R \\ [\omega_R, 1] & \text{otherwise.} \end{cases} \tag{5}$$

In other words, model $\mathbf{P}^B$ fully reveals the state below ω_B but does not generate any information above ω_B; model $\mathbf{P}^R$ fully reveals the state below ω_R but does not generate any information above ω_R.

The problem of finding models $\mathbf{P}^B$ and $\mathbf{P}^R$ reduces to finding the thresholds ω_B and ω_R. We must have:

$$E[N(\tilde{\omega})|\tilde{\omega} \geq \omega_B] \geq 0 \tag{6}$$

$$E[N(\tilde{\omega})|\ \tilde{\omega} \geq \omega_B \text{ or } \tilde{\omega} \in [\omega_R, \omega_B) \text{ and } G(\tilde{\omega}) \geq 0] \geq 0. \tag{7}$$

Equation (6) ensures that when the regulator observes $\mathbf{P}^B$, he allows the bank to invest when $\omega \geq \omega_B$. Equation (7) ensures that when the regulator observes only $\mathbf{P}^R$, he allows the bank to invest when $\omega \geq \omega_R$. Note that in the second case, the regulator understands that the bank will use his second model $\mathbf{P}^B$ to decide whether to invest.

The bank's expected payoff is:

$$V(\omega_B, \omega_R) \equiv (1-q) \int_{\omega_R}^{\omega_B} 1_{\{\omega: G(\omega) > 0\}} G(\omega) dF(\omega) + \int_{\omega_B}^{1} G(\omega) dF(\omega). \tag{8}$$

In particular, from Lemma 2, we know that the bank always invests when $\omega \geq \omega_B$. If instead $\omega \in (\omega_R, \omega_B)$, the bank invests only if the regulator does not observe $\mathbf{P}^B$, which happens with probability $1-q$, and $G(\omega) \geq 0$.

Since the term inside the first integral in (8) is positive, the bank would like to set ω_R as low as possible subject to Equation (7). Hence, we can assume, without loss of generality, that

$$\omega_R = \bar{\omega}_R(\omega_B), \tag{9}$$

where $\bar{\omega}_R(\omega_B)$ is the lowest $\omega_R \in \Omega$ that satisfies Equation (7).[10]

As for ω_B, we proceed in two steps. First, we derive a necessary and sufficient condition for ω_B to be an equilibrium threshold. Then, we derive a closed-form solution.

Denote the lowest $\omega_B \in \Omega$ that satisfies Equation (6) by $\bar{\omega}_B$. Clearly, we must have

$$\omega_B \geq \bar{\omega}_B. \tag{10}$$

In addition, for every $\omega'_B \geq \bar{\omega}_B$, we must have

$$V(\omega_B, \bar{\omega}_R(\omega_B)) \geq V(\omega'_B, \bar{\omega}_R(\omega_B)). \tag{11}$$

Equation (11) rules out a deviation in which the bank chooses model $\mathbf{P}^B$ with threshold ω'_B instead of ω_B, while keeping model $\mathbf{P}^R$ unchanged with threshold $\omega^{-}{}_R(\omega_B)$. It turns out that ruling out this deviation is not only a necessary equilibrium condition but is also a suffi cient condition. Formally:

Lemma 3 $\omega_B \in \Omega$ *is an equilibrium threshold if and only if* $\omega_B \geq \bar{\omega}_B$ *and Equation (11) holds for every* $\omega'_B \in \Omega$ *such that* $\omega'_B \geq \bar{\omega}_B$*.*

We can reduce the set of potential equilibrium thresholds even further. Let K denote the set of left corners of intervals in which an unregulated bank invests (e.g., the left corners of the red intervals in Figure 1). In equilibrium, we must have $\omega_B \in K \cup \{\bar{\omega}_B\}$. To see why, note that if ω_B lies inside an interval in which the bank does not want to invest, the bank can increase its payoff by increasing ω_B. If ω_B lies inside an interval in which the bank wants to invest, the bank can increase its payoff by reducing ω_B, but ω_B cannot fall below $\bar{\omega}_B$.

[10] Any $\omega_R \in [\bar{\omega}_R(\omega_B), \hat{\omega}_R(\omega_B)]$ will give the same outcome, where $\hat{\omega}_R(\omega_B)$ denotes the lowest $\omega_R \in \Omega$ that satisfies both Equation (7) and $G(\omega_R) \geq 0$. Any other ω_R will give a worse outcome for the bank.

Formally, define a set $\Omega_0 \subset \Omega$ as follows:

$$\Omega_0 = \begin{cases} \{\omega \in K : \omega \geq \bar{\omega}_B\} \cup \{\bar{\omega}_B\} & \text{if } G(\bar{\omega}_B) \geq 0 \\ \{\omega \in K : \omega \geq \bar{\omega}_B\} & \text{otherwise} \end{cases}$$

We can replace Lemma 3 with the following:

Lemma 4 *$\omega_B \in \Omega$ is an equilibrium threshold if and only if $\omega_B \in \Omega_0$ and Equation (11) holds for every $\omega'_B \in \Omega_0$.*

We can use Lemma 4 to derive a closed-form solution for ω_B as a function of q. To obtain intuition, we start with the simple case in which Ω_0 contains only two thresholds ω_1 and ω_2 ($\omega_1 > \omega_2$). Let

$$\rho(\omega_1, \omega_2) \equiv \frac{|\int_{\omega_2}^{\omega_1} 1_{\{\omega:G(\omega)<0\}} G(\omega) dF(\omega)|}{\int_{\omega_2}^{\omega_1} 1_{\{\omega:G(\omega)>0\}} G(\omega) dF(\omega).} \tag{12}$$

Proposition 1 *If Ω_0 contains only two thresholds $\omega_1 > \omega_2$, then:*

1. *If $q < \rho(\omega_1, \omega_2)$, $\omega_B = \omega_1$ is a unique equilibrium threshold.*
2. *If $q = \rho(\omega_1, \omega_2)$, both $\omega_B = \omega_1$ and $\omega_B = \omega_2$ are equilibrium thresholds.*
3. *If $q > \rho(\omega_1, \omega_2)$, $\omega_B = \omega_2$ is a unique equilibrium threshold.*

Intuitively, choosing a higher threshold ω_B corresponds to producing more information. This is beneficial for the bank because the bank can make better investment decisions for its equity holders. However, producing more information can also be costly for the bank because the regulator can use the extra pieces of information to ban investment in some states in which the bank wants to invest. This tradeoff for the bank is captured by the ratio ρ in Equation (12). Specifi cally, the numerator refl ectsthe gain from choosing the higher threshold $\omega_B = \omega_1$,; namely, the bank avoids investment when $\omega \in (\omega_2, \omega_1)$ and $G(\omega) < 0$. The denominator (times q) captures the cost; namely, the regulator bans investment when $\omega \in (\omega_2, \omega_1)$ and $G(\omega) > 0$. Since the expected cost is increasing in q, the bank prefers the higher threshold only if q is suffi ciently low.

Proposition 1 extends to the general case in which Ω_0 contains n thresholds $\omega_1 > \omega_2 > ... > \omega_n$. In particular, the equilibrium threshold ω_B can be described by a step function, which is decreasing in q. Consistent with Part 2 in Proposition 1, at the corners of steps, there could be more than one equilibrium threshold. In what follows, we focus on the equilibrium threshold ω_B that is most preferred by the regulator: namely, the equilibrium with the highest threshold. This equilibrium is also weakly preferred by the bank.[11] We let $\omega_B(q)$ stand for the equilibrium threshold ω_B for a given q.

Theorem 1 *There exist $\delta_1, \delta_2, ..., \delta_m \in \Omega_0$ such that*

$$\omega_B(q) = \begin{cases} \delta_1 & \text{if } q \in [0, \bar{q}_1] \\ \delta_2 & \text{if } q \in (\bar{q}_1, \bar{q}_2] \\ \vdots & \\ \delta_m & \text{if } q \in (\bar{q}_{m-1}, 1], \end{cases} \tag{13}$$

where $\bar{q}_i = \rho(\delta_i, \delta_{i+1})$ for $i \in \{1, 2, ..., m-1\}$. Moreover, $\delta_1 = \omega_1 > \delta_2 > ... > \delta_m$.

The proof of Theorem 1 (in the Appendix) fully defines the number of steps m and the values for $\delta_1, \delta_2, ..., \delta_m$. Note that $\bar{q}_i$ is the bank's net gain from producing more information by moving from a lower threshold $\omega_B = \delta_{i-1}$ to a higher threshold $\omega_B = \delta_i$. The theorem captures the intuition that when the regulator monitors more, the bank produces less information.[12]

6.2 Optimal q

The regulator's payoff from choosing a probability of monitoring q is:

$$u(q) \equiv (1-q)\int_{\bar{\omega}_R(\omega_B(q))}^{\omega_B(q)} 1_{\{\omega: G(\omega)\geq 0\}} N(\omega) dF(\omega) + \int_{\omega_B(q)}^{1} N(\omega) dF(\omega). \tag{14}$$

The regulator's payoff is similar to the bank's [Equation (8)], but instead of $G(\omega)$, we have $N(\omega)$. In equilibrium, the regulator chooses $q \in [0, 1]$ to maximize (14).

Since $\omega_B(q)$ is a left-continuous step function and the first integral in (14) is nonpositive, the regulator's problem has a solution that lies at the (right) corners of

[11] In particular, when ω_B is higher, the bank can satisfy Equation (7) by setting a weakly lower ω_R.

[12] Consistent with Blackwell (1951), less information here means less information that is relevant to investment decisions.

the intervals that define the step function in Theorem 1. That is, there is a solution $q \in \{\bar{q}_1, ..., \bar{q}_{m-1}, 1\}$. In general, a solution to the regulator's problem cannot lie inside an interval $(\bar{q}_{i-1}, \bar{q}_i)$, except for the case in which $q = 1$ is optimal and the first integral in (14) equals zero. In this case, any $q \in (\bar{q}_{m-1}, 1]$ is optimal. Formally, let $\bar{q}_0 \equiv 0$. Then:

Lemma 5 *If the step function in Theorem 1 has only one step, then $u(1) \geq u(q)$ for every $q \in [0, 1)$. If the step function has $m \geq 2$ steps, then:*

1. $u(\bar{q}_1) > u(0)$ and $u(\bar{q}_i) > u(q)$ for every $q \in (\bar{q}_{i-1}, \bar{q}_i)$ and $i \in \{1, ..., m-1\}$.

2. $u(1) \geq u(q)$ for every $q \in (\bar{q}_{m-1}, 1)$.

Moreover, all inequalities above are strict if $\omega_B(1) > \bar{\omega}_B$ and $G(\omega) > 0$ for some $\omega < \omega_B(1)$.

We illustrate our main results for the case in which $\Omega_0 = \{\omega_1, \omega_2\}$, where $\omega_1 > \omega_2 > \bar{\omega}_B$. If $\rho(\omega_1, \omega_2) \geq 1$, then consistent with Proposition 1, the step function in Theorem 1 has only one step: $\omega_B(q) = \omega_1$ for every $q \in [0, 1]$. If $\rho(\omega_1, \omega_2) < 1$, the step function has two steps:

$$\omega_B(q) = \begin{cases} \omega_1 & \text{if } q \leq \rho(\omega_1, \omega_2) \\ \omega_2 & \text{if } q > \rho(\omega_1, \omega_2) \end{cases} \tag{15}$$

Hence, if $\rho(\omega_1, \omega_2) \geq 1$, it is optimal to set $q = 1$. If $\rho(\omega_1, \omega_2) < 1$, setting $q = 1$ is optimal only if $u(1) \geq u(\rho(\omega_1, \omega_2))$. This condition reduces to $\rho(\omega_1, \omega_2) \leq \hat{q}$, where

$$\hat{q} \equiv \frac{|\int_{\bar{\omega}_R(\omega_1)}^{\omega_2} 1_{\{\omega:G(\omega)\geq 0\}} N(\omega) dF(\omega)| - |\int_{\omega_2}^{\omega_1} 1_{\{\omega:G(\omega)<0\}} N(\omega) dF(\omega)|}{|\int_{\bar{\omega}_R(\omega_1)}^{\omega_2} 1_{\{\omega:G(\omega)\geq 0\}} N(\omega) dF(\omega)| + |\int_{\omega_2}^{\omega_1} 1_{\{\omega:G(\omega)\geq 0\}} N(\omega) dF(\omega)|}. \tag{16}$$

Observe that $\hat{q} < 1$, and that for $\hat{q} > 0$ to hold, we must have $G(\omega) > 0$ for some $\omega < \omega_2$. We obtain that:

Proposition 2 *If $\Omega_0 = \{\omega_1, \omega_2\}$, where $\omega_1 > \omega_2 > \bar{\omega}_B$, then:*

1. If $\rho(\omega_1, \omega_2) > \hat{q}$, the regulator sets $q = \min\{\rho(\omega_1, \omega_2), 1\}$, and the bank responds by choosing $\omega_B = \omega_1$.

2. If $\rho(\omega_1, \omega_2) < \hat{q}$, the regulator sets $q = 1$, and the bank responds by choosing $\omega_B = \omega_2$.

3. If $\rho(\omega_1, \omega_2) = \hat{q}$, both $q = \rho(\omega_1, \omega_2)$ and $q = 1$ are optimal.

Recall that $\rho(\omega_1, \omega_2)$ is the bank gain (relative to cost) from producing information (i.e., choosing $\omega_B = \omega_1$ rather than $\omega_B = \omega_2$). Proposition 2 says that it is optimal to set $q = 1$ when the bank's gain from producing information is either sufficiently high or sufficiently low. In the first case, the bank produces information even if $q = 1$. Setting $q = 1$ is uniquely optimal because any other q induces the bank to produce the same amount of information for itself but reveal less information to the regulator. In the second case, when the bank's gain from producing information is low, the regulator can still induce the bank to produce information, but only if he precommits to a very low level of monitoring. But then, the regulator cannot make much use of the information that the bank produces and is better off monitoring more, even though this induces the bank to produce less information overall.

The intuition above extends to the general case. If the step function in Theorem 1 has only one step, $q = 1$ is optimal. This case happens if $\rho(\omega_1, \omega) \geq 1$ for every $\omega \in \Omega_0$ such that $\omega < \omega_1$. If the step function has $m > 1$ steps, $q = 1$ is optimal only if $u(1) \geq u(\bar{q}_i)$ for every $i \in \{1, 2, ..., m-1\}$. The last condition reduces to

$$\bar{q}_i \leq \frac{|\int_{\bar{\omega}_R(\delta_i)}^{\delta_m} 1_{\{\omega:G(\omega)\geq 0\}} N(\omega) dF(\omega)| - |\int_{\delta_m}^{\delta_i} 1_{\{\omega:G(\omega)<0\}} N(\omega) dF(\omega)|}{|\int_{\bar{\omega}_R(\delta_i)}^{\delta_m} 1_{\{\omega:G(\omega)\geq 0\}} N(\omega) dF(\omega)| + |\int_{\delta_m}^{\delta_i} 1_{\{\omega:G(\omega)\geq 0\}} N(\omega) dF(\omega)|} \tag{17}$$

for every $i \in \{1, 2, ..., m-1\}$.

Theorem 2 *If $\Omega_0 = \{\omega_1\}$, then $q = 1$ is optimal and is uniquely optimal if $\omega_1 > \bar{\omega}_B$ and $G(\omega) > 0$ for some $\omega < \omega_1$. If Ω_0 contains at $n \geq 2$ thresholds, then:*

1. *$q = 1$ is optimal if and only if either the step function in Theorem 1 has only one step (i.e., $\min_{\omega\in\Omega_0:\omega<\omega_1} \rho(\omega_1, \omega) \geq 1$) or the step function has $m > 1$ steps and Equation (17) holds for every $i \in \{1, 2, ..., m-1\}$.*

2. *$q = 1$ is uniquely optimal if either $\min_{\omega\in\Omega_0:\omega<\omega_1} \rho(\omega_1, \omega) \geq 1$ or $\omega_n > \bar{\omega}_B$ and Equation (17) holds with strict inequalities for every $i \in \{1, 2, ..., m-1\}$.*

7 Comparative statics

We showed that it is optimal to set a high level of monitoring when the bank's gain from producing information is either sufficiently high or sufficiently low. We can use this insight to derive some comparative statics. We illustrate for the case in which $\Omega_0 = \{\omega_1, \omega_2\}$, where $\omega_1 > \omega_2 > \bar{\omega}_B$. We focus on the case $\hat{q} > 0$, in which changing the gain from producing information $\rho(\omega_1, \omega_2)$ has a nonmonotone effect on the optimal q.

We start our discussion on comparative statics with a formal analysis when information production is costly. Suppose the bank incurs a cost $z > 0$ if it produces information on states above ω_2. That is, the bank incurs the cost if it chooses a model that includes a set in $(\omega_2, 1]$. As before, we can focus, without loss of generality, on models that take the simple form as in Equations (4) and (5). So, the bank's expected payoff is $V(\omega_B, \omega_R) - z1_{\{\omega_B > \omega_2\}}$.

The cost z does not change the set Ω_0 from which the bank chooses ω_B, but it changes the bank's private gain from producing information.[13] Specifically, for Proposition 1 to hold, instead of $\rho(\omega_1, \omega_2)$, we must have $\zeta(z)$, where

$$\zeta(z) = \frac{|\int_{\omega_2}^{\omega_1} 1_{\{\omega: G(\omega)<0\}} G(\omega) dF(\omega)| - z}{\int_{\omega_2}^{\omega_1} 1_{\{\omega: G(\omega)\geq 0\}} G(\omega) dF(\omega).} \tag{18}$$

Intuitively, the cost z reduces the benefits from producing information, and this is reflected in the numerator of Equation (18).

Using the logic from the previous section, the optimal q is as follows: If $\zeta(z) \geq 1$, it is optimal to set $q = 1$, and the bank responds by choosing $\omega_B = \omega_1$. This case happens when z is suffi ciently low. If instead $\zeta(z) < 1$, the regulator compares

between choosing $q = 1$ and choosing $q = \zeta(z)$. In the first case, the bank responds by choosing $\omega_B = \omega_2$, and in the second case, the bank responds by choosing $\omega_B = \omega_1$. The regulator also takes into account the social cost z of producing information. So, setting $q = 1$ is optimal only if $u(1) \geq u(\zeta(z)) - z$. Since $u(\zeta(z)) - z$ is decreasing in z, the last condition holds only if z is sufficiently high.

Proposition 3 *There exists $z_2 \in \mathbb{R}$ such that:*

[13] Note that a different specification for the cost function (e.g., assuming that the cost is $z\omega_B$) could potentially change the set Ω_0. Although the main force and intuition we identify here will still be valid, some of the derivations and arguments may be more convoluted (see also Footnote 15).

1. If $z < z_2$, the regulator sets $q = \min\{1, \zeta(z)\}$ and the bank responds by choosing $\omega_B = \omega_1$. In this range, q is decreasing in z.

2. If $z > z_2$, the regulator sets $q = 1$, and the bank responds by choosing $\omega_B = \omega_2$.

3. If $z = z_2$, the regulator is indifferent between setting $q = 1$ and setting $q = \zeta(z)$.

Part 1 captures the intuition that if the cost of producing information is sufficiently low, the regulator can induce the bank to produce information while still maintaining a relatively high level of monitoring. Part 2 captures the intuition that as the cost of producing information increases, it becomes too costly, or even impossible, for the regulator to induce the bank to produce information. In this case, it is optimal to set a high level of monitoring, even though the bank will produce less information.

In the remainder of this section, we discuss other comparative statics that might be of interest. In particular, model parameters, such as D and X, affect the bank's gain from investing, $G(\omega)$, which, in turn, affects $\rho(\omega_1, \omega_2)$.

For example, when the bank's debt level D increases, or when the value of its existing asset $v(\omega)$ falls (e.g., when the curve $v(\omega)$ shifts downward), the bank's gain from default increases, and so, $G(\omega)$ increases. An increase in project cash flows x also increases $G(\omega)$, as the project's NPV increases. In all three cases, $\rho(\omega_1, \omega_2)$ becomes lower, which follows directly from Equation (12), noting that the numerator decreases and the denominator increases. Intuitively, when the bank's gain from investment increases, the bank has less incentive to produce information because the cost that the regulator will use the information to ban investment becomes more significant, while at the same time, the benefit from producing information to avoid investment in negative NPV projects becomes less significant.

Hence, for a given $q > 0$, an increase in D, an increase in x, or a reduction in $v(\omega)$, lead the bank to produce less information.[14]

As for the optimal q, under some regularity conditions,[15] the fact that model para-

[14] Note that these parameter changes could also affect the set $\{\omega : G(\omega) = 0\}$, and hence, the set Ω_0 from which the bank chooses ω_B. This effect also works in the same direction, reducing the amount of information that the bank produces. For example, when D increases, the thresholds ω_1 and ω_2 become lower.

[15] Such conditions are needed because, as noted earlier, the set Ω_0 could change. This could affect

meters affect the bank' sgain from producing information leads to similar implications as in Proposition 2.

Specifically, with respect to the bank debt level, when D is very low, the bank has strong incentives to produce information and, therefore, the regulator chooses a high level of monitoring without much perverse effect. As D increases, the regulator needs to lower the level of monitoring to induce information production. Finally, when D is suffi ciently high, the regulator moves back to full monitoring as information production is very diffi cult to induce.

As for the asset value $v(\omega)$, the predictions are opposite to those related to D. When asset values are high, the regulator monitors extensively, as the bank has strong incentives to produce information. When asset values are moderate, the regulator monitors less to induce the bank to produce more information. When asset values are low, the regulator monitors extensively, but the bank produces less information.

Finally, as for the project cash flows x, when x is low, the regulator monitors extensively, and the bank produces a lot of information. When x is medium, the regulator monitors less to induce the higher level of information production. When x is sufficiently high, the regulator monitors extensively, but the bank produces less information.

8 Public information

Suppose everyone is endowed with some model $\hat{\mathbf{P}}$. In other words, model $\hat{\mathbf{P}}$ is public information. For example, we can think of $\hat{\mathbf{P}}$ as existing rules in place that are common knowledge. So, in step 3 in the sequence of events, the bank observes $P^B(\omega)$ and $\hat{P}(\omega)$. As for the regulator, with probability q, he observes $P^B(\omega)$ and $\hat{P}(\omega)$, and with probability $1-q$, he observes $P^R(\omega)$ and $\hat{P}(\omega)$.

Let ϕ_1 and ϕ_2 be the corners of the information set in $\hat{\mathbf{P}}$ that contains the state $\frac{1}{x}$. That is, $\phi_1 = \inf\{\omega \in \Omega : \frac{1}{x} \in \hat{P}(\omega)\}$ and $\phi_2 = \sup\{\omega \in \Omega : \frac{1}{x} \in \hat{P}(\omega)\}$. For

the regulator's gain from implementing a higher level of information production versus a lower level of information production. Moreover, with respect to x, we should also take into account the direct effect of x on the regulator's payoff.

any $\mathbf{P}^B$ and $\mathbf{P}^R$, if $\omega < \phi_1$, the regulator will ban investment. Similarly, if $\omega > \phi_2$, the regulator will allow investment and the bank will invest. Hence, the models that the bank chooses affect the outcome only when $\omega \in (\phi_1, \phi_2)$ (and potentially at the corners ϕ_1 and ϕ_2).

The problem reduces to finding thresholds ω_B and ω_R, as in the previous section, but instead of $\bar{\omega}_B$ and $\bar{\omega}_R(\omega_B)$, we now have $\breve{\omega}_B$ and $\breve{\omega}_R(\omega_B)$, which are defined as follows: $\breve{\omega}_B$ is the lowest $\omega_B \in [\phi_1, \phi_2]$ that satisfies

$$E[N(\tilde{\omega})|\tilde{\omega} \in [\omega_B, \phi_2)] \geq 0, \tag{19}$$

and $\breve{\omega}_R(\omega_B)$ is the lowest $\omega_R \in [\phi_1, \phi_2]$ that satisfies

$$E[N(\tilde{\omega})|\tilde{\omega} \in [\omega_B, \phi_2) \text{ or } \tilde{\omega} \in [\omega_R, \omega_B) \text{ and } G(\omega) \geq 0] \geq 0. \tag{20}$$

An interesting question is how the optimal q changes when ϕ_1 or ϕ_2 change. We illustrate for the case in which $\Omega_0 = \{\omega_1, \omega_2\}$, where $\omega_1 > \omega_2 > \bar{\omega}_B$. We focus on the more interesting case in which $\rho(\omega_1, \omega_2) < \hat{q}$.[16]

Consider first the case $\phi_2 = 1$.

Proposition 4 *If $\phi_2 = 1$ and Ω_0 contains only two thresholds $\omega_1 > \omega_2 > \bar{\omega}_B$, such that $\hat{q} > \rho(\omega_1, \omega_2)$, then there exists $\hat{\phi} \in (\bar{\omega}_R(\omega_1), \omega_2)$ such that:*

1. If $\phi_1 < \hat{\phi}$, the regulator sets $q = 1$, and the bank chooses $\omega_B = \omega_R = \omega_2$.

2. If $\phi_1 \in (\hat{\phi}, \omega_2]$, the regulator set $q = \rho(\omega_1, \omega_2)$, and the bank chooses $\omega_B = \omega_1$ and $\omega_R < \omega_1$.

3. If $\phi_1 \in (\omega_2, \omega_1]$, the regulator sets $q = \min\{1, \rho(\omega_1, \phi_1)\}$, which is increasing in ϕ_1, and the bank chooses $\omega_B = \omega_1$ and $\omega_R < \omega_1$.

4. If $\phi_1 \in (\omega_1, \frac{1}{x})$, any q is optimal, and the bank choice of models is irrelevant.

(When $\phi_1 = \hat{\phi}$, there are two equilibria: the equilibrium from Part 1 and the equilibrium from Part 2.)

[16] If $\rho(\omega_1, \omega_2) \geq 1$, the regulator chooses $q = 1$ and the bank chooses $\omega_B = \omega_R = a_2$, independently of the value of ϕ_1. If $\rho(\omega_1, \omega_2) \in (\hat{q}, 1)$, the first case in the proposition below is absent, and the second case holds for every $\phi_1 < \omega_2$.

Proposition 4 illustrates two forces that push q in different directions. When the public model is more informative (ϕ_1 increases), the benefit from monitoring the bank is reduced because the regulator can use the public model to ban investment. This can lead to a lower q. However, it can also become easier to induce the bank to produce information, because if the regulator uses the public model to ban investment, the bank gains less from not producing information. This can lead to a higher q. The first force is present in Part 2 in Proposition 4. The second force is present in Part 3.

Figure 2 illustrates. In particular, if $\phi_1 < \omega_2$, an increase in ϕ_1 affects q because it increases $\breve{\omega}_R(\omega_1)$, and hence, the regulator's payoff $u(\rho(\omega_1, \omega_2))$. Since there is no effect on $u(1)$, the regulator switches from $q = 1$ to $q = (\omega_1, \omega_2)$. If $\phi_1 > \omega_2$, then $\breve{\omega}_B = \phi_1$, and the set Ω_0 becomes $\{\phi_1, \omega_1\}$. In this case, the regulator sets $q = \rho(\phi_1, \omega_1)$, which is increasing in ϕ_1.

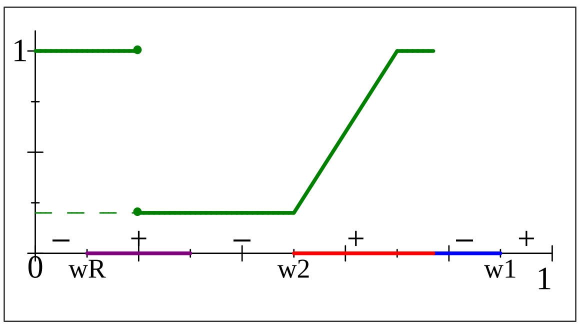

Figure 2. Optimal q as a function of ϕ_1

Finally, reducing ϕ_2 has a similar effect to increasing ϕ_1. Specifically, when ϕ_2 is sufficiently large so that $\breve{\omega}_B < \omega_2$, reducing ϕ_2 reduces $\breve{\omega}_R(\omega_1)$, and hence leads to a lower q. When ϕ_2 is lower, so that $\breve{\omega}_B > \omega_2$, reducing ϕ_2 increases $\breve{\omega}_B$, and hence, leads to a higher q.

9 Applications

The insights from our model can be applied in other settings, such as when regulators use bank internal risk models to set minimum capital requirements. Suppose capital can be either high or low, and suppose that instead of the endogenous functions $N(\omega)$

and $G(\omega)$, we have some exogenous function, which represent the benefit from low capital versus high capital to the regulator and to the bank, respectively. Our main results continue to hold as long as $G(\omega) \geq N(\omega)$ for every $\omega \in \Omega$, and $N(\omega)$ is increasing in ω. That is, the bank's gain from having a lower amount of capital is larger than the regulator's gain, and the regulator's gain is increasing in the state. ($N(\omega)$ and $G(\omega)$ can take both positive values and negative values.) The insights from our model can apply in this setting if we relabel "investment" to mean having low capital requirements. So, allowing investment means that the regulator allows the bank to have low capital, while banning investment means that the regulator requires high capital. Our model suggests the following:

1. When existing rules (e.g., risk weights) measure risk imperfectly, the regulator might gain from relying on bank internal models.

2. It might be optimal to allow the bank to produce two different models: one for regulation and one for its own trading.

3. When existing rules measure risk more precisely, the bank produces more information.

4. The relationship between the quality of existing rules and the optimal level of monitoring is nonmonotone.

Our framework can also be applied in corporate governance. The optimality of delegation of authority to the management has been studied in the context of information transmission and incorporation of private information into decision making. Our model provides new insights on the benefits of curbing boards' and shareholders' power to create shareholder value.

10 Concluding remarks

We analyze a situation in which a regulator relies on information that a bank produces to regulate the bank. We show that monitoring induces a tradeoff. By monitoring the bank, the regulator obtains information, which could be used to the regulator's advantage. However, a higher level of monitoring could induce the bank to produce less information overall. Solving for the optimal level of monitoring, we show that, in general, the optimal level of monitoring should be high when the bank's private gain from producing information is either sufficiently high or sufficiently low. We use this result to derive comparative statics as to how the optimal level of monitoring varies with respect to model parameter, such as the bank's level of debt. We also analyze the case in which some public information already exists. We show that when public models are more precise, banks produce more information, but the regulator may end up monitoring more.

One can think of our framework as a persuasion game in which the bank produces two signals. The first signal is used to persuade the regulator to delegate authority to the bank. The second signal is used to make better investment decisions, once authority is delegated. As is standard in the Bayesian persuasion literature, we assumed that the bank has full control in choosing the information technology, and, as such, the regulator cannot (or chooses not to) force the bank to choose a specific information technology. This could be due to the fact that the bank's ability to produce information is private information.

Two assumptions are crucial for our results. First, the regulator can commit to a prespecified monitoring intensity, but he cannot commit to contracts that bind him to allow investment when it is inefficient to do so given his information ex post. Second, we need a restriction on the information technology. In particular, the bank cannot produce information partitions that pool together high and low states while excluding the states in the middle. If we maintained only the first assumption but not the second, the bank would not need to generate a second signal, and the optimal signal could be obtained along the lines of Kamenica and Gentzkow (2011).

In practice, the regulator's commitment to a monitoring technology often arises via various mechanisms and rules that dictate what the regulator and banks can or cannot do. Our findings lend support in favor of simple policy rules, as opposed to complicated recipes that try to get all available information from banks to fine tune regulation. Our main trade-off continues to hold in dynamic settings, or in settings where full commitment is not feasible, as long as the regulator cannot completely adjust the monitoring technology perfectly ex post.

Our framework can be extended in several directions. One possible path is to allow the regulator to choose whether to rely on banks'internal models. The main tradeoff we identified will continue to hold in such an extension. However, a richer strategy space will enable us to address other issues in bank regulation, such as the impact of using banks' models on banks' risk taking behavior. Another path is to allow the regulator to use banks'internal models to impose capital requirements coupled with restrictions on investment decisions or endogenous investment decisions by the bank. Such an extension will enable us to make more complete policy prescriptions.

References

[1] Aghion, Philippe, and Jean Tirole (1997). Formal and Real Authority in Organizations. *Journal of Political Economy*, 105, 1–29.

[2] Blackwell, David (1951). Comparison of Experiments. *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, University of California Press, Berkeley, CA, 93-102.

[3] Begley, Taylor A., Amiyatosh K. Purnanandam, and Kuncheng (K.C.) Zheng (2016). The Strategic Under-Reporting of Bank Risk. Ross School of Business Paper 1260.

[4] Behn, Markus, Rainer Haselmann, and Vikrant Vig (2014). The Limits of Model-Based Regulation. SAFE Working Paper 75.

[5] Blum, Jürg M. (2008). Why 'Basel II' May Need a Leverage Ratio Restriction. *Journal of Banking and Finance*, 32, 1699–1707.

[6] Burkart, Mike, Denis Gromb, and Fausto Panunzi (1997). Large Shareholders, Monitoring, and the Value of the Firm. *Quarterly Journal of Economics*, 112, 693–728.

[7] Chakraborty, Archishman and Bilge Yilmaz (2014). Authority, Consensus and Governance. Working paper.

[8] Cohn, Jonathan, Uday Rajan, and Günter Strobl (2013). Credit Ratings: Strategic Issuer Disclosure and Optimal Screening. Working Paper.

[9] Crémer, Jacques (1995). Arm's Length Relationships. *Quarterly Journal of Economics*, 110, 275–295.

[10] Dessein, Wouter (2002). Authority and Communication in Organizations. *Review of Economic Studies*, 69, 811–838.

[11] Firestone, Simon, and Marcelo Rezende (2016). Are Banks' Internal Risk Parameters Consistent? Evidence from Syndicated Loans. *Journal of Financial Services Research,* 50, 211–242.

[12] Grenadier, Steven, Andrey Malenko, and Nadya Malenko (2016). Timing Decisions in Organizations: Communication and Authority in a Dynamic Environment. *American Economic Review,* 106, 2552–2581.

[13] Harris, Milton, and Artur Raviv (2005). Allocation of Decision-Making Authority. *Review of Finance*, 9, 353–383.

[14] Harris, Milton, and Artur Raviv (2008). A Theory of Board Control and Size. *Review of Financial Studies*, 21, 1797–1832.

[15] Harris, Milton, and Artur Raviv (2010). Control of Corporate Decisions: Shareholders vs. Management. *Review of Financial Studies*, 23, 4115–4147.

[16] Kamenica, Emir, and Matthew Gentzkow (2011). Bayesian Persuasion. *American Economic Review*, 101, 2590–2615.

[17] Mariathasan, Mike, and Ouarda Merrouche (2014). The Manipulation of Basel Risk-Weights. *Journal of Financial Intermediation*, 23, 300–321.

[18] Marshall, David A., and Edward S. Prescott (2006). State-Contingent Bank Regulation with Unobserved Actions and Unobserved Characteristics. *Journal of Economic Dynamics and Control*, 30, 2015–2049.

[19] Plosser, Matthew C., and João A. C. Santos (2014). Banks' Incentives and the Quality of Internal Risk Models. Federal Reserve Bank of New York Staff Reports 704.

[20] Polinsky, A. Mitchell, and Steven Shavell (2012). Mandatory Versus Voluntary Disclosure of Product Risks. *Journal of Law, Economics, and Organization*, 28, 360–379.

[21] Prescott, Edward S. (2004). Auditing and Bank Capital Regulation. Federal Reserve Bank of Richmond *Economic Quarterly*, 90, 47–63.

[22] Shavell, Steven (1994). Acquisition and Disclosure of Information Prior to Sale. *RAND Journal of Economics*, 25, 20–36.

Appendix

Proof of Lemma 1. We first show that if $G(\omega) \geq 0$ then either (i) or (ii) holds. To see that, note that if (i) does not hold, then $N(\omega) < 0$. So, to satisfy $G(\omega) \geq 0$, we must have $D > v(\omega)$. Hence, $N(\omega) + (1-\omega)[D - v(\omega)] \geq 0$, which reduces to $v(\omega) \leq D + \frac{N(\omega)}{1-\omega}$. Hence, (ii) holds. Next, we show that if (i) or (ii) holds, then $G(\omega) \geq 0$. Clearly, if (i) holds, then $N(\omega) \geq 0$, and so, $G(\omega) \geq 0$. If (ii) holds, then $N(\omega) + (1-\omega)[D - v(\omega)] \geq 0$, and so $N(\omega) + (1-\omega)\max[D - v(\omega), 0] \geq 0$. Hence, $G(\omega) \geq 0$.

Proof of Lemma 2. Consider an equilibrium in which the bank chooses models $\mathbf{P}^B$ and $\mathbf{P}^R$. The equilibrium outcome can be described by a pair of functions $I_B, I_R : \Omega \to \{0,1\}$, where for $\gamma \in \{B, R\}$, $I_\gamma(\omega) = 1$ if and only if the bank invests when the state is ω and the regulator observes $P^\gamma(\omega)$. We use $\inf P^\gamma(\frac{1}{x})$ to denote the highest $\omega \in \Omega$ that is less than or equal to every state in $P^\gamma(\frac{1}{x})$. Similarly, $\sup P^\gamma(\frac{1}{x})$ is the lowest $\omega \in \Omega$ that is greater than or equal to every state in $P^\gamma(\frac{1}{x})$. Since $\mathbf{P}^B$ is at least as informative as $\mathbf{P}^R$, $\sup P^R(\frac{1}{x}) \geq \sup P^B(\frac{1}{x})$ and $\inf P^R(\frac{1}{x}) \leq \inf P^B(\frac{1}{x})$.

Because each partition element is convex, it follows for $\gamma \in \{B, R\}$ that if $\omega > \sup P^\gamma(\frac{1}{x})$ and $\omega' \in P^\gamma(\omega)$, then $\omega' > \frac{1}{x}$. Similarly, if $\omega < \inf P^\gamma(\frac{1}{x})$ and $\omega' \in P^\gamma(\omega)$, then $\omega' < \frac{1}{x}$. Hence, for $\gamma \in \{B, R\}$, when the regulator observes $P^\gamma(\omega)$, then if $\omega > \sup P^\gamma(\frac{1}{x})$, both the bank and regulator know that the project has positive NPV, and if $\omega < \inf P^\gamma(\frac{1}{x})$, both the bank and regulator know that the project has negative NPV. Hence, the equilibrium outcome must be such that if $\omega > \sup P^\gamma(\frac{1}{x})$, the regulator allows investment and the bank invests. In contrast, if $\omega < \inf P^\gamma(\frac{1}{x})$, investment does not take place, because if the bank invests, the regulator expects to end up with a negative payoff and is better off banning investment.

Let

$$\omega_B = \begin{cases} \inf P^B(\frac{1}{x}) & \text{if } I_B(\frac{1}{x}) = 1 \\ \sup P^B(\frac{1}{x}) & \text{otherwise.} \end{cases} \tag{A-1}$$

It follows that

$$I_B(\omega) = \begin{cases} 1 & \text{if } \omega > \omega_B \\ 0 & \text{if } \omega < \omega_{B.} \end{cases} \tag{A-2}$$

Finally, suppose $\omega \in (\inf P^R(\frac{1}{x}), \sup P^R(\frac{1}{x}))$ and the regulator observes $P^R(\omega)$. Let $\omega_R = \inf P^R(\frac{1}{x})$ if the regulator allows investment, and $\omega_R = \sup P^R(\frac{1}{x})$, otherwise. If the regulator does not allow investment, we must have $\omega_B = \sup P^R(\frac{1}{x})$ (and, hence, $\omega_R = \omega_B$) because if $\omega_B < \sup P^R(\frac{1}{x})$, the bank could increase its payoff by choosing model $\hat{\mathbf{P}}^R = \mathbf{P}^B$ instead of $\mathbf{P}^R$, whereas if $\omega_B > \sup P^R(\frac{1}{x})$, we would obtain a contradiction $\omega_B > \sup P^R(\frac{1}{x}) \geq \sup P^B(\frac{1}{x})$. Now suppose the regulator allows investment. In this case, $\omega_R \leq \omega_B$, and the bank invests if and only if $E[G(\tilde{\omega})|\tilde{\omega} \in P^B(\omega)] \geq 0$. So, if $\omega > \omega_B$, the bank invests. If, instead, $\omega \in (\inf P^R(\frac{1}{x}), \omega_B)$, the equilibrium outcome must be such that the bank invests if $G(\omega) > 0$ but not if $G(\omega) < 0$. To see why, observe that if the bank does not invest at some ω such that $G(\omega) > 0$, then $E[G(\tilde{\omega})|\tilde{\omega} \in P^B(\omega)] < 0$, and there must be an interval that contains ω, such that the bank does not invest in that interval. Moreover, since G is continuous, that interval contains a positive measure of states with $G(\tilde{\omega}) > 0$. Similarly, if the bank invests at some ω such that $G(\omega) < 0$, there must be an interval that contains ω, such that the bank invests in that interval, and that interval contains a positive measure of states with $G(\tilde{\omega}) < 0$. Hence, if, to the contrary, the equilibrium outcome is such that the bank invests if $G(\omega) < 0$ or does not invest if $G(\omega) > 0$, we obtain a contradiction because the bank could increase its payoff by choosing model $\hat{\mathbf{P}}^B$ instead of $\mathbf{P}^B$, where $\hat{\mathbf{P}}^B$ is defined as follows:

$$\hat{P}^B(\omega) = \begin{cases} P^B(\omega) & \text{if } \omega > \omega_B \\ \omega & \text{otherwise.} \end{cases} \tag{A-3}$$

In particular, if the regulator observes $P^B(\omega)$, investment will not take place when $\omega < \omega_B$, and the bank is not worse off by revealing the exact state $\omega < \omega_B$. If, however, the regulator does not observe $P^B(\omega)$, the bank is better off learning the exact state and investing according to its ideal investment rule.

Hence, we showed that

$$I_R(\omega) = \begin{cases} 1 & \text{if } \omega > \omega_B \text{ or if } \omega \in (\omega_R, \omega_B) \text{ and } G(\omega) > 0 \\ 0 & \text{if } \omega < \omega_R \text{ or if } \omega \in (\omega_R, \omega_B) \text{ and } G(\omega) < 0 \end{cases} \tag{A-4}$$

This completes the proof.

Proof of Lemma 3. Consider an equilibrium in which the bank chooses models $\mathbf{P}^B$ and $\mathbf{P}^R$ with thresholds ω_B and ω_R, as in Equations (4) and (5). To satisfy Equation (6), we must have $\omega_B \geq \bar{\omega}_B$. In addition, as explained in the text, Equation (11) must hold for every $\omega'_B \in \Omega$ such that $\omega'_B \geq \bar{\omega}_B$. Note that if the bank deviates by choosing $\hat{\mathbf{P}}^B \neq \mathbf{P}^B$, and the regulator observes only model $\mathbf{P}^R$, he continues to believe that the other model is $\mathbf{P}^B$, and so he continues to allow investment when $\omega \geq \omega_R$.

It remains to show the other direction. Suppose $\omega_B \geq \bar{\omega}_B$ and Equation (11) holds for every $\omega'_B \in \Omega$ such that $\omega'_B \geq \bar{\omega}_B$. We show that there is an equilibrium in which the bank chooses models $\mathbf{P}^B$ and $\mathbf{P}^R$ with thresholds ω_B and $\omega_R = \bar{\omega}_R(\omega_B)$. In such an equilibrium, the bank's payoff is $V(\omega_B, \bar{\omega}_R(\omega_B))$. We need to show that the bank cannot increase its payoff by choosing different models. Consider a deviation in which the bank chooses models $\hat{\mathbf{P}}^B$ and $\hat{\mathbf{P}}^R$, which do not necessarily take the simple form in Equations (4) and (5). Assign out-of-equilibrium beliefs such that when the regulator observes $\hat{\mathbf{P}}^R \neq \mathbf{P}^R$, he believes that the other model is $\mathbf{P}^B$. Hence, upon observing $\hat{\mathbf{P}}^R(\omega)$, the regulator anticipates that the bank will invest if and only if $E[G(\tilde{\omega})|\tilde{\omega} \in \mathbf{P}^B(\omega)] \geq 0$. Following the logic of Lemma 2, we can show that $\hat{\omega}_B, \hat{\omega}_R \in \Omega$ exist such that the outcome of this deviation is as follows. When the regulator observes model $\hat{\mathbf{P}}^B(\omega)$, the bank invests if $\omega > \hat{\omega}_B$ but not if $\omega < \hat{\omega}_B$. And when the regulator observes models $\hat{\mathbf{P}}^R(\omega)$, the regulator allows the bank to invest when $\omega > \hat{\omega}_R$, and the bank does not invest when $\omega < \hat{\omega}_R$. For the regulator to allow investment when $\omega > \hat{\omega}_B$, it must be the case that $\hat{\omega}_B \geq \bar{\omega}_B$. For the regulator to allow investment when $\omega > \hat{\omega}_R$, it must be the case that $E[N(\tilde{\omega})|\tilde{\omega} > \omega_B \text{ or } \tilde{\omega} \in [\hat{\omega}_R, \omega_B) \text{ and } G(\tilde{\omega}) \geq 0]$. Hence, by the definition of $\bar{\omega}_R$, we must have $\hat{\omega}_R \geq \bar{\omega}_R(\omega_B)$. Consequently, the bank's payoff from the deviation is at most $V(\hat{\omega}_B, \bar{\omega}_R(\omega_B))$. Hence, from our starting assumption that Equation (11) holds for every $\omega'_B \in \Omega$ such that $\omega'_B \geq \bar{\omega}_B$, the payoff from the deviation is at most $V(\omega_B, \bar{\omega}_R(\omega_B))$. Hence, choosing $\mathbf{P}^B$ and $\mathbf{P}^R$ is an equilibrium.

Proof of Lemma 4. Recall there exist a finite set of numbers $b_1 > a_1 > \ldots >$

$b_l > a_l$, such that $G(\omega) \geq 0$ if and only if $\omega \in \cup_{i=1}^{l}[a_i, b_i]$. Hence, $K = \{a_1, ..., a_l\}$.

We first show that for every $\omega \notin \Omega_0$ and $\omega_R \in \Omega$, such that $\omega \geq \bar{\omega}_B \geq \omega_R$, there exists $\omega' \in \Omega_0$, such that $V(\omega, \omega_R) < V(\omega', \omega_R)$. Consider $\omega \notin \Omega_0$ and $\omega_R \in \Omega$, such that $\omega \geq \bar{\omega}_B \geq \omega_R$. There exists $i \in \{1, ..., l\}$ such that either $\omega \in (a_i, b_i]$ or $\omega \in (b_{i+1}, a_i)$. If $\omega \in (a_i, b_i]$, let $\omega' = \max\{a_i, \bar{\omega}_B\}$. If $\omega \in (b_{i+1}, a_i)$, let $\omega' = a_i$. Then $\omega' \in \Omega_0$ and $V(\omega, \omega_R) < V(\omega', \omega_R)$.

We use the observation above to prove Lemma 4. Suppose $\omega_B \in \Omega$ is an equilibrium threshold. From Lemma 3, $\omega_B \geq \bar{\omega}_B$ and Equation (11) holds for every $\omega'_B \in \Omega$ such that $\omega'_B \geq \bar{\omega}_B$. Hence, Equation (11) holds under the weaker condition $\omega'_B \in \Omega_0$. Moreover, we must have $\omega_B \in \Omega_0$ because otherwise, the observation above would imply that there exists $\omega' \in \Omega_0$, such that $V(\omega_B, \bar{\omega}_R(\omega_B)) < V(\omega', \bar{\omega}_R(\omega_B))$, which contradicts Lemma 3.

Now suppose $\omega_B \in \Omega_0$ and Equation (11) holds for every $\omega'_B \in \Omega_0$. We show that Equation (11) also holds for every $\omega'_B \notin \Omega_0$ such that $\omega'_B \geq \bar{\omega}_B$, and hence, by Lemma 3, ω_B is an equilibrium threshold. Suppose to the contrary that $\omega'_B \notin \Omega_0$ exists such that $\omega'_B \geq \bar{\omega}_B$ and $V(\omega_B, \bar{\omega}_R(\omega_B)) < V(\omega'_B, \bar{\omega}_R(\omega_B))$. From the observation above, $\omega' \in \Omega_0$ exists, such that $V(\omega'_B, \bar{\omega}_R(\omega_B)) < V(\omega', \bar{\omega}_R(\omega_B))$. Hence, $V(\omega_B, \bar{\omega}_R(\omega_B)) < V(\omega', \bar{\omega}_R(\omega_B))$, which contradicts the starting assumption that Equation (11) holds for every $\omega'_B \in \Omega_0$.

Lemma A-1 *For any $\omega_R \in \Omega$ and $\omega, \omega' \in \Omega_0$, such that $\omega > \omega'$, the following holds:*

1. *If $q < \rho(\omega, \omega')$, then $V(\omega, \omega_R) > V(\omega', \omega_R)$.*
2. *If $q = \rho(\omega, \omega')$, then $V(\omega, \omega_R) = V(\omega', \omega_R)$.*
3. *If $q > \rho(\omega, \omega')$, then $V(\omega, \omega_R) < V(\omega', \omega_R)$.*

Proof of Lemma A-1. The proof applies to a more general case in which there is a cost $z \geq 0$, as in Section 7. Consider $\omega_R \in \Omega$ and $\omega, \omega' \in \Omega_0$, such that $\omega > \omega'$.

Observe that $V(\omega, \omega_R) - z > V(\omega', \omega_R)$ is equivalent to

$$(1-q)\int_{\omega_R}^{\omega} 1_{\{\omega:G(\omega)>0\}} G(\omega)dF(\omega) + \int_{\omega}^{1} G(\omega)dF(\omega) - z \tag{A-5}$$
$$> \; (1-q)\int_{\omega_R}^{\omega'} 1_{\{\omega:G(\omega)>0\}} G(\omega)dF(\omega) + \int_{\omega'}^{1} G(\omega)dF(\omega).$$

After rearranging terms, (A-5) reduces to

$$(1-q)\int_{\omega'}^{\omega} 1_{\{\omega:G(\omega)>0\}} G(\omega)dF(\omega) > \int_{\omega'}^{\omega} G(\omega)dF(\omega) + z, \tag{A-6}$$

which reduces to

$$\int_{\omega'}^{\omega} 1_{\{\omega:G(\omega)\geq 0\}} G(\omega)dF(\omega) - \int_{\omega'}^{\omega} G(\omega)dF(\omega) - z > q\int_{\omega'}^{\omega} 1_{\{\omega:G(\omega)>0\}} G(\omega)dF(\omega). \tag{A-7}$$

Since $\omega, \omega' \in \Omega_0$ and $\omega > \omega'$, the integral on the right-hand-side of (A-7) is positive. Hence, (A-7) reduces to

$$q < \frac{-\int_{\omega'}^{\omega} 1_{\{\omega:G(\omega)<0\}} G(\omega)dF(\omega) - z}{\int_{\omega'}^{\omega} 1_{\{\omega:G(\omega)>0\}} G(\omega)dF(\omega)} = \frac{|\int_{\omega'}^{\omega} 1_{\{\omega:G(\omega)<0\}} G(\omega)dF(\omega)| - z}{\int_{\omega'}^{\omega} 1_{\{\omega:G(\omega)>0\}} G(\omega)dF(\omega)}. \tag{A-8}$$

Hence, we proved part 1. Parts 2 and 3 follow in a similar fashion.

Proof of Proposition 1. Since Ω_0 contains only two thresholds $\omega_1 > \omega_2$, we know from Lemma 4 that for ω_B to be an equilibrium threshold, we must have $\omega_B \in \{\omega_1, \omega_2\}$. To see why part 1 is true, suppose $q < \rho(\omega_1, \omega_2)$. From Lemma A-1, $V(\omega_1, \bar{\omega}_R(\omega_1)) > V(\omega_2, \bar{\omega}_R(\omega_1))$ and $V(\omega_1, \bar{\omega}_R(\omega_2)) > V(\omega_2, \bar{\omega}_R(\omega_2))$. Hence, by Lemma 4, ω_1 is an equilibrium threshold, whereas ω_2 is not an equilibrium threshold. Parts 2 and 3 follow similarly.

Proof of Theorem 1. We construct the step function $\omega_B(q)$ as follows. Let $\bar{q}_0 = 0$, $\delta_1 = \omega_1$, and for integers $i \geq 1$, define recursively:

$$\bar{q}_i = \begin{cases} \min\{1, \min_{\omega\in\Omega_0\cap[0,\delta_i)} \rho(\delta_i, \omega)\} & \text{if } \delta_i > \omega_n \\ 1 & \text{otherwise.} \end{cases} \tag{A-9}$$

$$\delta_{i+1} = \begin{cases} \min\{\omega \in \Omega_0 \cap [0, \delta_i) : \rho(\delta_i, \omega) = \bar{q}_i\} & \text{if } \bar{q}_i < 1 \\ \delta_i & \text{otherwise.} \end{cases} \tag{A-10}$$

Let $m = \min\{i : \bar{q}_i = 1\}$.

From the definition of m, $\bar{q}_i < 1$ for every $i < m$. Hence, from Equation (A-10), $\delta_1 > \delta_2 > ... > \delta_m$ and

$$\bar{q}_i = \rho(\delta_i, \delta_{i+1}) \text{ for every } i < m. \tag{A-11}$$

In addition, from Equation (A-9), $\delta_i > \omega_n$ for every $i < m$.

We show that $\omega_B(0) = \delta_1$, and that for $i \in \{1, ..., m\}$, $\omega_B(q) = \delta_i$ if $q \in (\bar{q}_{i-1}, \bar{q}_i]$. The proof is by induction. For $i = 1$, we know from the definition of $\bar{q}_1$ that $\bar{q}_1 \leq \min_{\omega \in \Omega_0 \cap [0,\omega_1)} \rho(\omega_1, \omega)$. Hence, when $q \leq \bar{q}_1$, we know from Lemma A-1 that $V(\omega_1, \bar{\omega}_R(\omega_1)) \geq V(\omega, \bar{\omega}_R(\omega_1))$ for every $\omega \in \Omega_0 \cap [0, \omega_1)$, and so by Lemma 4, ω_1 is an equilibrium threshold. Since in case of multiple equilibria, we focus on the one with the highest threshold, it follows that $\omega_B(q) = \omega_1$ when $q \in [0, \bar{q}_1]$.

Now suppose $i < m$ and $\omega_B(q) = \delta_i$ if $q \in (\bar{q}_{i-1}, \bar{q}_i]$. We show that $\omega_B(q) = \delta_{i+1}$ if $q \in (\bar{q}_i, \bar{q}_{i+1}]$. There are a few steps:

1. Form Lemma A-1, it follows that if $V(\omega_B, \bar{\omega}_R(\omega_B)) \geq V(\omega'_B, \bar{\omega}_R(\omega_B))$, then $V(\omega_B, \omega_R) \geq V(\omega'_B, \omega_R)$ for every $\omega_R \in \Omega$. Hence, we can rewrite Lemma 4 as follows: $\omega_B \in \Omega$ is an equilibrium threshold if and only if $\omega_B \in \Omega_0$ and $V(\omega_B, \omega_R) \geq V(\omega', \omega_R)$ for every $\omega' \in \Omega_0$ and $\omega_R \in \Omega$.

2. We show that if $q = \bar{q}_i$, then $V(\delta_{i+1}, \omega_R) \geq V(\omega', \omega_R)$ for every $\omega' \in \Omega_0$ and $\omega_R \in \Omega$. In other words, if $q = \bar{q}_i$, then δ_{i+1} weakly dominates any other threshold candidate. This follows from the following two observations. First, if $q = \bar{q}_i$, we know from the induction assumption that δ_i is an equilibrium threshold, and so, from Step 1, $V(\delta_i, \omega_R) \geq V(\omega', \omega_R)$ for every $\omega' \in \Omega_0$ and $\omega_R \in \Omega$. Second, since $\bar{q}_i = \rho(\delta_i, \delta_{i+1})$, it follows from Lemma A-1 that $V(\delta_{i+1}, \omega_R) = V(\delta_i, \omega_R)$ for every $\omega_R \in \Omega$.

3. Now we show that if $q > \bar{q}_i$, then δ_{i+1} strictly dominates any other threshold candidate that is greater than δ_{i+1}. It follows from Step 2 and Lemma A-1 that $\bar{q}_i \geq \rho(\omega', \delta_{i+1})$ for every $\omega' \in \Omega_0 \cap (\delta_{i+1}, 1]$. It then follows from A-1 that if $q > \bar{q}_i$, then $V(\delta_{i+1}, \omega_R) > V(\omega', \omega_R)$ for every $\omega' \in \Omega_0 \cap (\delta_{i+1}, 1]$ and $\omega_R \in \Omega$.

4. It also follows from Step 2 and Lemma A-1 that $\bar{q}_i \leq \rho(\delta_{i+1}, \omega)$ for every $\omega \in \Omega_0 \cap [0, \delta_{i+i})$.

5. If $\delta_{i+1} = \omega_n$, then $\bar{q}_{i+1} = 1$, and from Steps 1 and 3, δ_{i+1} is a unique equilibrium when $q \in (\bar{q}_i, \bar{q}_{i+1}]$. So, the proof is complete.

6. If $\delta_{i+1} > \omega_n$, then $\bar{q}_{i+1} \leq \min_{\omega \in \Omega_0 \cap [0,\delta_{j+1})} \rho(\delta_{i+1}, \omega)$. Hence, from Lemma A-1, if $q \leq \bar{q}_{i+1}$, then $V(\delta_{i+1}, \omega_R) \geq V(\omega', \omega_R)$ for every $\omega' \in \Omega_0 \cap [0, \delta_{i+1})$ and $\omega_R \in \Omega$. In other words, δ_{i+1} weakly dominates any other threshold candidate that is smaller than δ_{i+1}. Note that if $\bar{q}_{i+1} \neq 1$, then $\bar{q}_{i+1} = \rho(\delta_{i+1}, \delta_{i+2}) \geq \bar{q}_i$, where the last inequality follows from Step 4.

7. It follows from Steps 1, 3, and 6, that if $q \in (\bar{q}_i, \bar{q}_{i+1}]$, δ_{i+1} is an equilibrium and any other $\omega_B \in \Omega_0 \cap (\delta_{i+1}, 1]$ is not an equilibrium. Since in case of multiple equilibria we focus on the one with the highest threshold, it follows that $\omega_B(q) = \delta_{i+1}$ if $q \in (\bar{q}_i, \bar{q}_{i+1}]$.

Proof of Lemma 5. For use below, denote $\bar{q}_m = 1$. From Theorem 1, $\omega_B(\bar{q}_1) = \omega_B(0)$, and for every $i \in \{1, ..., m\}$, $\omega_B(\bar{q}_i) = \omega_B(q)$ if $q \in (\bar{q}_{i-1}, \bar{q}_i)$. In addition, $\omega_B(\bar{q}_i) > \omega_B(\bar{q}_{i-1})$. Also note that since $G(\omega) \geq 0$ for every $\omega \geq \frac{1}{x}$ (Lemma 1), then $\omega_B(\bar{q}_1) \leq \frac{1}{x}$, and so $N(\omega) < 0$ for every $\omega < \omega_B(\bar{q}_1)$. Hence, if $m \geq 2$ and $q \leq \bar{q}_{m-1}$, then $\omega_B(q) > \omega_B(1) \geq \bar{\omega}_B$, and so, the first integral in (14) satisfies $\int_{\bar{\omega}_R(\omega_B(q))}^{\omega_B(q)} 1_{\{\omega:G(\omega)\geq 0\}} N(\omega) dF(\omega) \leq \int_{\omega_B(1)}^{\omega_B(q)} 1_{\{\omega:G(\omega)\geq 0\}} N(\omega) dF(\omega) < 0$. Hence, Part 1 follows. If $m = 1$ or if $m \geq 2$ and $q \in (\bar{q}_{m-1}, 1)$, the first integral in (14) is negative only if $\omega_B(1) > \bar{\omega}_B$ and $G(\omega) > 0$ for some $\omega < \omega_B(1)$. Otherwise, that integral equals zero. Hence, the rest of the lemma follows.

Proof of Proposition 2. Suppose $\Omega_0 = \{\omega_1, \omega_2\}$, where $\omega_1 > \omega_2 > \bar{\omega}_B$. Then in Theorem 1, $\delta_1 = \omega_1$ and $\bar{q}_1 = \min\{1, \rho(\omega_1, \omega_2)\}$. If $\rho(\omega_1, \omega_2) \geq 1$, then $m = 1$, and by Lemma 5, $q = 1$ is uniquely optimal. If $\rho(\omega_1, \omega_2) < 1$, then $\delta_2 = \omega_2$, $\bar{q}_2 = 1$, and $m = 2$. So from Lemma 5, there is a solution in $\{\rho(\omega_1, \omega_2), 1\}$. If $u(1) > u(\rho(\omega_1, \omega_2))$, $q = 1$ is optimal. This condition reduces to $\rho(\omega_1, \omega_2) < \hat{q}$. (The proof of Theorem 2 contains more details.) Since $\rho(\omega_1, \omega_2) \geq 0$, the last condition can hold only if $\hat{q} > 0$. In turn, $\hat{q} > 0$ holds only if $| \int_{\bar{\omega}_R(\omega_1)}^{\omega_2} 1_{\{\omega:G(\omega)\geq 0\}} N(\omega) dF(\omega)|$, which holds only if $G(\omega) > 0$ for some $\omega < \omega_2$. Hence, by Lemma 5, $q = 1$ is uniquely optimal when $\rho(\omega_1, \omega_2) < \hat{q}$. If $\rho(\omega_1, \omega_2) \in (\hat{q}, 1)$, then $u(1) < u(\rho(\omega_1, \omega_2))$, and by Lemma 5,

$q = \rho(\omega_1, \omega_2)$ is uniquely optimal. Combining the results above, we obtain Parts 1 and 2. Part 3 follows easily.

Proof of Theorem 2. The result for $\Omega_0 = \{\omega_1\}$ follows immediately from Lemma 5. Now suppose Ω_0 contains $n \geq 2$ thresholds. Then, in Theorem 1, $\delta_1 = \omega_1$ and $\bar{q}_1 = \min\{1, \min_{\omega \in \Omega_0 \cap [0, \delta_i)} \rho(\delta_i, \omega)\}$. If $\min_{\omega \in \Omega_0 : \omega < \omega_1} \rho(\omega_1, \omega) \geq 1$, then $m = 1$, and by Lemma 5, $q = 1$ is uniquely optimal. The rest of this proof focuses on the case $m = 2$.

From Lemma 5, $q = 1$ is optimal if and only if $u(1) \geq u(\bar{q}_i)$ for every $i \in \{1, ..., m-1\}$. So, to prove the first part, we need to show that for every $i \in \{1, ..., m-1\}$, $u(1) \geq u(\bar{q}_i)$ reduces to Equation (17). The details are as follows. Observe that $u(1) = \int_{\delta_m}^{1} N(\omega) dF(\omega)$, and for $i \in \{1, ..., m-1\}$, $u(\bar{q}_i) = (1 - \bar{q}_i) \int_{\bar{\omega}_R(\delta_i)}^{\delta_i} 1_{\{\omega : G(\omega) \geq 0\}} N(\omega) dF(\omega) + \int_{\delta_i}^{1} N(\omega) dF(\omega)$. Hence, after rearranging terms, $u(1) \geq u(\bar{q}_i)$ reduces to

$$\int_{\delta_m}^{\delta_i} N(\omega) dF(\omega) \geq (1 - \bar{q}_i) \int_{\bar{\omega}_R(\delta_i)}^{\delta_i} 1_{\{\omega : G(\omega) \geq 0\}} N(\omega) dF(\omega). \tag{A-12}$$

Since $\delta_m < \delta_{m-1} < ... < \delta_1 \leq \frac{1}{x}$, it follows that $N(\omega) < 0$ when $\omega < \delta_i$. Hence, the integrals on both sides of Equation (A-12) are negative. Hence, Equation (A-12) reduces to

$$1 - \bar{q}_i \geq \frac{\int_{\delta_m}^{\delta_i} N(\omega) dF(\omega)}{\int_{\bar{\omega}_R(\delta_i)}^{\delta_i} 1_{\{\omega : G(\omega) \geq 0\}} N(\omega) dF(\omega)} = \frac{|\int_{\delta_m}^{\delta_i} N(\omega) dF(\omega)|}{|\int_{\bar{\omega}_R(\delta_i)}^{\delta_i} 1_{\{\omega : G(\omega) \geq 0\}} N(\omega) dF(\omega)|}, \tag{A-13}$$

or equivalently,

$$\begin{aligned} \bar{q}_i &\leq \frac{|\int_{\bar{\omega}_R(\delta_i)}^{\delta_i} 1_{\{\omega : G(\omega) \geq 0\}} N(\omega) dF(\omega)| - |\int_{\delta_m}^{\delta_i} N(\omega) dF(\omega)|}{|\int_{\bar{\omega}_R(\delta_i)}^{\delta_i} 1_{\{\omega : G(\omega) \geq 0\}} N(\omega) dF(\omega)|} \\ &= \frac{|\int_{\bar{\omega}_R(\delta_i)}^{\delta_m} 1_{\{\omega : G(\omega) \geq 0\}} N(\omega) dF(\omega)| - |\int_{\delta_m}^{\delta_i} 1_{\{\omega : G(\omega) < 0\}} N(\omega) dF(\omega)|}{|\int_{\bar{\omega}_R(\delta_i)}^{\delta_m} 1_{\{\omega : G(\omega) \geq 0\}} N(\omega) dF(\omega)| + |\int_{\delta_m}^{\delta_i} 1_{\{\omega : G(\omega) \geq 0\}} N(\omega) dF(\omega)|}. \end{aligned} \tag{A-14}$$

Hence, we showed that $u(1) \geq u(\bar{q}_i)$ holds if and only if Equation (17) holds, which completes the proof of Part 1.

To show the second part, observe that for every $i \in \{1, ..., m-1\}$, $u(1) > u(\bar{q}_i)$ if and only if Equation (17) holds with strict inequalities. Hence, from Lemma 5, it remains to show that $G(\omega) > 0$ for some $\omega < \delta_m$. This follows because if Equation (17) holds with strict inequalities when $i = m-1$, then $|\int_{\bar{\omega}_R(\delta_{m-1})}^{\delta_m} 1_{\{\omega:G(\omega)\geq 0\}} N(\omega)dF(\omega)| > 0$, which implies that $G(\omega) > 0$ for some $\omega < \delta_m$.

Proof of Proposition 3. The logic from Section 6.1 applies for the case under consideration, but for Proposition 1 to hold, we must have $\zeta(z)$ instead of $\rho(\omega_1, \omega_2)$. This follows from the proof of Lemma A-1. Observe that $\zeta(z)$ is decreasing in z. Let z_1 be the unique z that satisfies $\zeta(z) = 1$, and let z_2 be the unique z that satisfies

$$\int_{\omega_2}^{1} N(\omega)dF(\omega) = (1-\zeta(z)) \int_{\bar{\omega}_R(\omega_1)}^{\omega_1} 1_{\{\omega:G(\omega)\geq 0\}} N(\omega)dF(\omega) + \int_{\omega_1}^{1} N(\omega)dF(\omega) - z. \tag{A-15}$$

A unique z_2 exists because $\int_{\bar{\omega}_R(\omega_1)}^{\omega_1} 1_{\{\omega:G(\omega)\geq 0\}} N(\omega)dF(\omega) < 0$, and so the right-hand-side in Equation (A-15) is decreasing in z. Moreover, $z_2 > z_1$. To see that, it is sufficient to show that when $\zeta(z) = 1$, the right-hand-side in Equation (A-15) is greater than the left-hand side. This follows because from Equation (18), $\zeta(z) = 1$ implies that

$$\begin{aligned} -z &> -|\int_{\omega_2}^{\omega_1} 1_{\{\omega:G(\omega)<0\}} G(\omega)dF(\omega)| = \int_{\omega_2}^{\omega_1} 1_{\{\omega:G(\omega)<0\}} G(\omega)dF(\omega) \\ &\geq \int_{\omega_2}^{\omega_1} 1_{\{\omega:G(\omega)<0\}} N(\omega)dF(\omega) \geq \int_{\omega_2}^{\omega_1} N(\omega)dF(\omega). \end{aligned} \tag{A-16}$$

If $z \leq z_1$, then $\zeta(z) \geq 1$, and so $\omega_B(q) = \omega_1$ for every $q \in [0,1]$. In this case, the regulator's payoff is $u(q) - z$, which is increasing in q. Hence, $q = 1$ is uniquely optimal.

If $z > z_1$, then $\zeta(z) < 1$, and so,

$$\omega_B(q) = \begin{cases} \omega_1 & \text{if } q \leq \zeta(z) \\ \omega_2 & \text{if } q > \zeta(z). \end{cases} \tag{A-17}$$

In this case, the regulator's payoff is $u(q) - z1_{\{\omega_B(q)=\omega_1\}}$. Hence, the left-hand side in Equation (A-15) is the regulator's payoff if $q = 1$, and the right-hand side is the regulator's payoff if $q = \zeta(z)$. Hence, when $z = z_2$, the regulator is indifferent between choosing $q = 1$ and choosing $q = \zeta(z)$. If $z \in (z_1, z_2)$, choosing $q = \zeta(z)$ is preferred to choosing $q = 1$. Similarly, if $z > z_2$, choosing $q = 1$ is preferred to choosing $q = \zeta(z)$.

Finally, from Lemma A-1 and the assumption that $\omega_2 > \bar{\omega}_B$ and $G(\omega) > 0$ for some $\omega < \omega_2$, it follows that any $q \notin \{\zeta(z), 1\}$ is suboptimal.

Proof of Proposition 4.

Parts 1 and 2. If $\phi_1 \leq \omega_2$, then

$$\omega_B(q) = \begin{cases} \omega_1 & \text{if } q \leq \rho(\omega_1, \omega_2) \\ \omega_2 & \text{otherwise} \end{cases} \tag{A-18}$$

and

$$u(q) = (1-q) \int_{\max\{\phi_1, \bar{\omega}_R(\omega_B(q))\}}^{\omega_B(q)} 1_{\{\omega: G(\omega) \geq 0\}} N(\omega) dF(\omega) + \int_{\omega_B(q)}^{1} N(\omega) dF(\omega). \tag{A-19}$$

From Proposition 2, we know that when $\phi_1 = 0$, the regulator sets $q = 1$. Since $\hat{q} > \rho(\omega_1, \omega_2) > 0$ and $\omega_2 > \bar{\omega}_B$, we know that $G(\omega) > 0$ for some $\omega < \bar{\omega}_B$. In addition, $G(\bar{\omega}_B) < 0$. Let $\omega_2' = \max\{\omega < \bar{\omega}_B : G(\omega) \geq 0\}$. When $\phi_1 = \omega_2'$, it is optimal to set $q = \rho(\omega_1, \omega_2)$ because

$$\begin{aligned} u(\rho(\omega_1, \omega_2)) &= (1-q) \int_{\omega_2}^{\omega_1} 1_{\{\omega: G(\omega) \geq 0\}} N(\omega) dF(\omega) + \int_{\omega_1}^{1} N(\omega) dF(\omega) \\ &> \int_{\omega_2}^{1} N(\omega) dF(\omega) = u(1). \end{aligned} \tag{A-20}$$

Since $u(q)$ is continuous and increasing in ϕ, and $u(1)$ does not depend on ϕ_1, there exists $\hat{\phi} \in (\bar{\omega}_R(\omega_1), \omega_2')$ such that when when $\phi_1 < \hat{\phi}$, the regulator sets $q = 1$, and when $\phi_1 \in (\hat{\phi}, \omega_2')$, the regulator sets $q = \rho(\omega_1, \omega_2)$.

Part 3. Let $\omega_1' = \max\{\omega < \omega_1 : G(\omega) \geq 0\}$. If $\phi_1 \in (\omega_2, \omega_1')$, the set Ω_0 changes to $\{\phi_1, \omega_1\}$. Hence,

$$\omega_B(q) = \begin{cases} \omega_1 & \text{if } q \leq \rho(\omega_1, \phi_1) \\ \phi_1 & \text{otherwise} \end{cases} \tag{A-21}$$

and

$$u(q) = (1-q) \int_{\phi_1}^{\omega_B(q)} 1_{\{\omega: G(\omega) \geq 0\}} N(\omega) dF(\omega) + \int_{\omega_B(q)}^{1} N(\omega) dF(\omega). \tag{A-22}$$

Since $u(\rho(\omega_1, \phi_1)) > u(1)$, it is (uniquely) optimal to choose $q = \rho(\omega_1, \phi_1)$. The bank responds by choosing $\omega_B = \omega_1$ and $\omega_R = \phi_1$.

Part 4. If $\phi_1 > \omega_1'$ the set Ω_0 includes only one state: $\max\{\omega_1, \phi_1\}$. Hence, $\omega_B(q) = \max\{\omega_1, \phi_1\}$, for every $q \in [0, 1]$. In this case, the first integral in Equation (A-22) equals zero, and any q is optimal.