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1 Introduction

This paper seeks to delimit the conditions so that market-based probabilities provide all the information required by the policymaker to arrive at the best decision possible. While there are several practical considerations regarding how to derive market-based probabilities from financial prices, the discussion here is confined to a theoretical analysis that assumes no impediment to obtaining the market-based probabilities.

The starting point of my analysis is a simple model in which policy must be set before the state of the world is realized.[1] The optimal policy requires knowledge of household preferences and the probability distribution across states, as well as the households' endowments and the net benefits of policy in each future state.

Can market-based probabilities provide such information to a policymaker? I allow households to trade a complete set of contingent claims, whose prices are perfectly observable. The policymaker then uses the underlying market-based probabilities to weigh the net benefits of policy across future states. Following Feldman et al. (2015), I provide sufficient conditions so that the policymaker is guaranteed to attain the optimal policy using only market-based probabilities—an equivalence result. The conditions are quite mild being as they are the same as those commonly used to guarantee that optimal policy can be characterized by the associated program's first-order conditions.

Unfortunately, the equivalence result is not as robust as it may appear. There are several instances in which the benchmark model implicitly assumes that the policymaker has access to additional knowledge that is not revealed by the market-based probabilities.

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[1]The model is a slightly simplified version of the framework presented in Feldman, Heinecke, Kocherlakota, and Schulhofer-Wohl (2015).

First, I provide a couple of examples that highlight the possibility that market-based probabilities may contain incomplete or no information at all. Second, I discuss the knowledge required to derive a net compensation function when policy has nonpecuniary effects. Whenever possible, I offer additional conditions to circumvent these problems.

The first concern is that market-based probabilities may not be informative. For example, an optimal policy may deliver full insurance to the households. Market prices are then equated across states, revealing no information. The equivalence result still applies: The only rational expectations equilibrium in which the policymaker relies on market-based probabilities is the optimal policy. Yet, it is left unspecified how the policymaker would actually arrive at the optimal policy given that the market-based probabilities are silent. A similar situation arises whenever the policy's net benefits are linear. A simple solution is to require that the policy's net benefits be strictly concave, though such an assumption would usually rule out policies that imply net transfers across states. I also show that the equivalence result does not carry over to the case of a discrete set of policies. This is perhaps not too surprising since market-based probabilities are informative of marginal effects, but the optimal policy is based on the average effect.

The second concern arises from nonpecuniary benefits from policy. It is possible to express the nonpecuniary benefits in terms of consumption using a "compensation" function. However, the equivalence result requires that the optimal policy is used as a baseline policy to compute the compensation function. Using another policy as a baseline creates a mismatch between the marginal utility of consumption as given by the market prices—evaluated at actual consumption—and the counterfactual marginal utility of consumption embedded in the compensation function, which includes the net benefits. Therefore, unless the optimal policy is known at the start, the policymaker would not be able to use market-based probabilities to arrive at the optimal policy decision.

This paper follows Kocherlakota (2013) and Feldman et al. (2015) very closely. They have argued for the use of market-based probabilities to weigh policy options, particularly those regarding monetary policy. Academics have also relied on market-based probabilities to inform their work in several fields. To mention a few examples in monetary economics, Bauer and Rudebusch (2013) and Bauer (2014) rely on market-based expectations of future short-term interest rates, observed from interest rates on government bonds or money market futures; Kitsul and Wright (2013) and Hilscher, Raviv, and Reis (2014) use market-based probabilities to assess inflation expectations. Bauer and Rudebusch (2015) instead caution against overrelying on market-based probabilities for policymaking, citing a variety of practical factors (e.g., market incompleteness) that can render financial prices hard to interpret.

The paper is organized as follows. Section 2 presents the benchmark model and reproduces the results from Feldman et al. (2015). Section 3 discusses the possibility that market-based probabilities are not informative; while Section 4 addresses the case of nonpecuniary benefits. Section 5 concludes.

2 Benchmark case

I present a simplified version here of the model in Feldman et al. (2015) and reproduce their main results. There are N possible states of the world, indexed by n, and with probability distribution $\pi_n > 0$. The sequence of actions are as follows:

1. Households trade a complete set of contingent claims on consumption;
2. Policymaker decides to take policy action $a \in A \subset \Re$;
3. The state of the world n is revealed; and
4. Payoffs are realized.

The payoff associated with state n and policy decision a is

$$U\left(y_n + B_n(a)\right)$$

where U is a utility function with the standard properties, namely differentiable, strictly increasing and strictly concave; y_n is a strictly positive endowment; and $B_n(a) : A \to \Re$ determines the net benefit of policy action $a \in A$ in state n. I assume B_n is bounded above by some large number and below by $-y_n$ to preserve the nonnegativity of consumption. Additional properties of $B_n(a)$, together with those of set A, will be discussed later.

2.1 Optimal policy

With full knowledge of all fundamentals in the economy, the policymaker can easily compute the optimal policy by solving

$$\sup_{a\in A} \sum_{n\in N} \pi_n U\left(y_n + B_n(a)\right). \tag{1}$$

Let a^* denote the optimal policy decision.

Assume A is a closed interval and $B_n(a)$ is differentiable everywhere. Then the following first-order condition

$$\sum_{n\in N} \pi_n U'\left(y_n + B_n(a^*)\right) B_n'(a^*) = 0 \tag{2}$$

is a necessary condition for optimality for interior solutions, which I assume from now on.[2]

As is commonly done, I further assume that $B_n(a)$ is weakly concave everywhere for all $n \in N$, $B_n''(a) \leq 0$, which implies that the necessary first-order condition (2) is also sufficient.

[2]It is also necessary to assume certain Inada conditions on B_n to guarantee interior solutions.

2.2 Using market-based probabilities

Consider now a policymaker who has limited knowledge about households' preferences, the probability distribution and the endowments across states. Can the policymaker turn to market-based probabilities to make the best policy decision?

To answer this question, we need to set up a game between households and the policymaker, following the timing of decisions previously described. It is important to formally distinguish the households' belief regarding the policy action to be undertaken by the policymaker, denoted $\hat{a}$, and the policymaker's strategy, mapping market-based probabilities into policy actions.

I start with the trading stage. Let $\hat{q}_n$ be the price of a claim to one unit of consumption in state n. Households are identical, so there is no trade in equilibrium and thus $\hat{q}_n$ can be easily priced as

$$\hat{q}_n = \pi_n U' \left(y_n + B_n(\hat{a}) \right). \tag{3}$$

Note that prices depend on the households' belief regarding policy. The underlying market-based probabilities can be recovered with a simple normalization,

$$q_n = \frac{\hat{q}_n}{\sum_{m \in N} \hat{q}_m}$$

that ensures that the market-based probabilities indeed add up to one. Let $Q \subset \Re^n_+$ be the set of price vectors spanned by the set of policy actions A.

Turning to the policy decision, I assume that the policymaker has full knowledge of net benefits function $B_n(a)$ for all $n \in N$, and market-based probabilities are perfectly observable. The policymaker strategy thus maps the vector of market-based probabilities, denoted q, into an action $a \in A$. The best response function, denoted $\alpha(q)$, is such that for all q in the N-simplex, the policy decision maximizes the weighted net benefits as follows:

$$\sum_n q_n B_n(\alpha(q)) \geq \sum_n q_n B_n(a) \tag{4}$$

for all $a \in A$. Note that the game requires a full strategy profile (i.e., a policy action for every possible vector of market-based probabilities, $\alpha : Q \to A$).

It is all set for the definition of an equilibrium. An equilibrium is a vector of market-based probabilities q, a best response function α, and a household policy belief $\hat{a} \in A$ such that

1. Market-based probabilities q satisfy (3) for each $n \in N$, given $\hat{a}$;
2. The best response function α satisfies (4) for all q in the N-simplex; and
3. The households' policy belief $\hat{a}$ satisfies rational expectations, $\hat{a} = \alpha(q)$.

Is the optimal policy an equilibrium? Is it the unique equilibrium? As shown in Feldman et al. (2015), it turns out that the response to both questions is yes under some mild conditions—more precisely, once we assume A is a closed interval and $B_n(a)$

is differentiable and weakly concave for all $n \in N$. To see this, note that a necessary condition for (4), given the above assumptions, is

$$\sum_n q_n B_n'(\alpha(q)) = 0,$$

for interior solutions, and the correspondent slacked conditions if $\alpha(q)$ is on the boundary of A. Thus, a necessary condition for a to be an equilibrium is that

$$\sum_n \pi_n U'\left(y_n + B_n(a)\right) B_n'(a) = 0. \tag{5}$$

Clearly, the optimal policy satisfies the necessary condition for equilibrium, for (2) and (5) are indeed the same equation. Differentiating (5) with respect to the policy action a, I obtain

$$\sum_n \pi_n U''\left(y_n + B_n(a)\right)\left(B_n'(a)\right)^2 + \sum_n \pi_n U'\left(y_n + B_n(a)\right) B_n''(a) < 0.$$

There is thus a single equilibrium that must be then equal to the optimal policy.[3]

At this point, the equivalence between solving the optimal policy problem and using market-based probabilities seems very robust, the conditions imposed on the net benefit function B_n being quite mild—as a matter of fact, being the same properties that guarantee that the first-order condition for optimal policy is both necessary and sufficient. In the next two sections, however, I will argue that the conditions for the equivalence result are more restrictive than they appear.

3 When are market prices informative?

3.1 Net transfers

Consider the following simple example. There are only two states, $N = 2$, with $y_1 < y_2$. For simplicity, I assume both states have the same probability. The policy action is a simple transfer of resources across states, $B_1(a) = a$ and $B_2(a) = -a$. The optimal policy, for any strictly concave utility function, is to deliver full insurance across states,

$$a^* = \frac{y_2 - y_1}{2}$$

such that consumption in both states is equated, $c_1 = c_2 = (y_1 + y_2)/2$.

The conditions for the equivalence result are readily met, and indeed the optimal policy is the unique equilibrium. A closer inspection, though, reveals that market prices

[3] Assuming the optimal policy is an interior point of A, there is no difficulty extending the reasoning to slacked first-order conditions for the best response function.

contain no information whatsoever. As the optimal policy equates the marginal utility of consumption across states, the prices of a consumption claim to each state are identical,

$$\frac{1}{2}U'\left(\frac{y_1+y_2}{2}\right) = \hat{q}_1 = \hat{q}_2 = \frac{1}{2}U'\left(\frac{y_1+y_2}{2}\right).$$

Market-based probabilities are also identical across states, thus being uninformative about either the likelihood or the willingness of the households to trade resources across states.[4] Indeed, if the households expect the optimal policy to be in place, $\hat{a} = a^*$, the policymaker is actually *indifferent* between any policy action $a \in A$, since

$$\sum_n q_n B_n(a) = 0$$

for any action $a \in A$.

Any choice other than the optimal policy a^* is, of course, not an equilibrium since it would violate rational expectations. Yet, it is left unspecified how the policymaker would arrive at the optimal policy decision at all unless first-hand information is available on the endowments or the households' expected policy, $a\hat{} = a^*$.[5] In other words, the equilibrium conditions implicitly endow the policymaker with a knowledge that is not revealed by market-based probabilities and whose absence was the motivation for the use of market-based probabilities in the first place. More generally, whenever the net benefit function is linear, the policymaker will actually be indifferent between any policy choice in equilibrium. It should be noted, though, that this situation can also arise in some particular cases for nonlinear net benefit functions because it can be readily checked in the previous example by setting $B_1(a) = a^2$ and $B_2(a) = -a^2$.

It is fair to call this situation unsatisfactory. Ideally, we would like to ensure that the best response by the policymaker is uniquely pinned down by the market-based probabilities. Perhaps the most straightforward way to do so is to require that

$$\alpha(q) = \arg\max_{a \in A} \sum_n q_n B_n(a)$$

has a unique solution for all q in the N-simplex. This can be accomplished by requiring that the net benefit function is *strictly* concave for all $n \in N$, $B_n''(a) < 0$.

Assuming strict rather than weak concavity for the net benefit function is not such a minor change as it may appear. Policies that implement net transfers across states do not easily satisfy strict concavity since, one presumes, there will be an inverse relationship between the net benefits in each state. For example, if the benefit of a policy in one state equals the cost of such a policy in another state, i.e., $B_n(a) = -B_m(a)$, then it is

[4] Note that this is true for any probability distribution or endowment values. Market-based probabilities are thus indeed silent regarding the structural parameters.

[5] Note that that the best response α is uniquely defined for any market-based probabilities $q_1 \neq q_2$. Yet, there is no continuity argument to anchor the equilibrium, since the best response will lie at the boundary of A for $q_1 \neq q_2$.

clearly not possible for the net benefit functions in both states to be strictly concave. The assumption of strict concavity, though, is not a necessary condition, and it is thus possible to find alternative conditions that guarantee the equivalence result between the optimal policy and the use of market-based probabilities.

3.2 Discrete policy choices

The equivalence result does not carry over easily to the case of a discrete set of policy alternatives. This is perhaps not too surprising, since financial prices capture the marginal effect of policy on consumption but the optimal policy problem compares instead with the policy's average effects.

Consider the classic example of a dam, where $a \in \{0, 1\}$ indexes whether the dam is built or not, and there are two states, $N = 2$, rain or shine. For simplicity, the net benefits of not building the dam, $a = 0$, are zero in both states, and the net benefit of the dam satisfies $B_1 > 0 > B_2$—that is, the dam is beneficial when it rains but its construction must be paid anyway even if the sun shines.[6] It is optimal to build the dam if

$$\pi_1 \left(U\left(y_1 + B_1\right) - U\left(y_1\right)\right) + \pi_2 \left(U\left(y_2 + B_2\right) - U\left(y_2\right)\right) > 0.$$

A policymaker observing the financial claims on the weather would build the dam if

$$\pi_1 U'\left(y_1 + B_1\right) B_1 + \pi_2 U'\left(y_2 + B_2\right) B_2 \geq 0.$$

Unfortunately, it is quite easy to find values such that building the dam is optimal but market-based probabilities lead the policymaker to the wrong decision. Assume that when it rains, it pours; in the absence of a dam, flooding ensues and households are substantially worse off when it rains than when the sun shines, $y(1) < y(2)$. The dam, though, provides irrigation and allows high-value crops to grow: Households are then better off when it rains than when the sun shines, $y(1) + B_1 > y(2) + B_2$. If households expect the dam to be built, the financial claim to consumption on a rainy day will be low and, in particular, below the financial claim on a sunny day, $q_1 < q_2$. Based on the financial prices, the policymaker may mistakenly believe that households do not value the dam enough.[7] The problem is that the average effect of the dam is positive, yet the marginal effect is negative—and market-based probabilities are informative only about the latter. In such cases, there may be no equilibrium in pure strategies, as the policymaker may actually lean toward building the dam if households expect the dam not to be built.

4 Nonpecuniary net benefits

The benchmark model assumes that the net benefits of a policy are computed in consumption terms:

$$c_n = y_n + B_n(a).$$

[6] Notation B_1 is a shortcut for $B_1(1)$ and so on.

[7] A numerical example in which this occurs is as follows: $y_1 = 1, y_2 = 5, \pi_1 = .05, B_1 = 9.5, B_2 = -.5$ and $U = \log$.

Many policies, though, will have nonpecuniary benefits. Consider a simple example of a policy with nonpecuniary, separable benefit function

$$U(y_n) + G_n(a),$$

where G_n is assumed to be strictly concave for all $n \in N$. Now, one can always translate the nonpecuniary benefits into a "compensation function" that captures the net benefits of the policy in consumption terms. The compensation function is defined as the amount of consumption an agent would demand to receive (or be willing to give up) for the implementation of a policy. It does require a baseline policy to compare with, which I denote $\bar{a} = A$. Then, define $B_n(a; \bar{a})$ for $a, \bar{a} \in A$ as the solution to

$$U\left(y_n + B_n(a; \bar{a})\right) + G_n\left(\bar{a}\right) = U\left(y_n\right) + G_n\left(a\right). \tag{6}$$

That is, $B_n(a; \bar{a})$ is the amount of consumption (positive or negative) that renders an agent indifferent between policies $\bar{a}$ and a.[8]

Does the equivalence result between the optimal policy and the market-based probabilities approach apply? The equivalence result is obtained only if the baseline policy used for the net benefit function is the exact optimal policy, $a^{-} = a^{*}$. However, this implies that the policymaker knows the optimal policy at the start. Without knowing how to set a^{-}, the policymaker would not be able to use market-based probabilities to arrive at the optimal policy decision.[9]

Let us develop the example (6). Assuming an interior solution, a necessary and sufficient condition for the optimal policy is

$$\sum_n \pi_n G_n'(a^*) = 0. \tag{7}$$

While the marginal utility of consumption is now irrelevant, the optimal policy still requires knowledge of the probability distribution across future events. Financial prices are now independent of the expected policy, since the latter has only nonpecuniary benefits,

$$\hat{q}_n = \pi_n U'(y_n).$$

Market-based probabilities are recovered as before. Note that households will not receive an actual transfer regarding policy, and thus the market-based probabilities are evaluated at the endowment value, y_n.

Consider a policymaker relying on market-based probabilities to decide policy by solving

$$\max_{a \in A} \sum_n q_n B_n(a; \bar{a}).$$

[8] There is no difference if the net benefit function in consumption terms is defined instead as an "equivalence" rather than compensation function—that is, as a transfer given in the event of policy a rather than $\bar{a}$.

[9] Solving for $B_n(a; \bar{a})$ also requires knowledge of U and y_n, in addition to the nonpecuniary net benefit function G_n. It does not require, though, knowledge of the probability distribution π_n.

The first-order condition associated with the previous problem is

$$\sum_n q_n \frac{\partial B_n(a;\bar{a})}{\partial a} = 0.$$

By the implicit function theorem, I differentiate (6) to obtain

$$\frac{\partial B_n(a;\bar{a})}{\partial a} = \frac{G_n'(a)}{U'\left(y_n + B_n(a;\bar{a})\right)}.$$

Substituting the market-based probabilities, I obtain

$$\sum_n \pi_n \frac{U'(y_n)}{U'\left(y_n + B_n(a;\bar{a})\right)} G_n'(a) = 0. \tag{8}$$

In general, the optimal policy a^* will not satisfy the necessary first-order condition for the policymaker—that is, (7) and (8) do not coincide.[10] It actually does so only if

$$B_n(a^*;\bar{a}) = 0$$

for all $n \in N$, which requires $\bar{a} = a^*$. In other words, the net benefit function must be constructed using the optimal policy as the baseline policy.[11]

As condition (8) makes clear, the problem here is a "mismatch" between the marginal utility of consumption as given by the market-based probabilities—evaluated at actual consumption—and the counterfactual marginal utility of consumption embedded in the net benefit function B_n, which treats the net benefit as actual consumption. The two coincide only when the household is not compensated, $B_n = 0$.

5 Conclusions

I should emphasize that market-based probabilities remain useful for the policy decisions even if market prices do not provide *all* the information required. Perhaps the correct approach is to consider a policymaker who updates the set of priors upon observing market prices. In some special situations, market prices will be very informative, and the resulting policy decision will approach the optimal policy. In most cases, prices may contain only some information, so the policymaker will reach an improved decision but not necessarily one close to the optimal policy. There may be instances in which market prices are misleading, but such cases are surely bound to be a minority.

[10] There is actually no guarantee that B_n will remain strictly concave, but I will assume so to focus on the information required to compute the net benefit function.

[11] There may be other, nongeneric instances where (7) and (8) coincide.

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