

# Financial Frictions in Production Networks\*

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preliminary and incomplete, comments are extremely welcome

## Abstract

We show that the input-output structure of an economy has important implications for the impact of financial frictions on aggregate activity. We first study two simple economies with different production networks: a horizontal and a vertical economy. We construct these economies so that in the absence of frictions they are allocatively equivalent. However, when firms face liquidity constraints, the two economies yield very different equilibrium properties. We find that (i) depending on the network structure the aggregate labor wedge can overstate the wedges faced by individual firms, (ii) for any allocation, the vertical economy requires greater amounts of aggregate liquidity, (iii) the vertical economy features a greater drop in output due to a tightening of constraints; we call this the “liquidity multiplier”, (iv) the network location of firms matter for the impact of their individual constraints on aggregate output, and (v) conventional measures of misallocation must be revised when considering complex production networks. We then solve the general model for arbitrary input-output structures as in Acemoglu et al (2012). We calibrate this model in order to match the input-output matrix of the U.S. economy and use this to explore the extent to which these interrelationships can explain the drop in output during the latest recession. We find a liquidity multiplier of around 3.8 for the U.S. economy. Furthermore, in order to generate the observed drop in output at the trough of the recession, our calibrated model would require a reduction in liquidity equal to only 1/6th the drop in liquidity required in a representative firm model.

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# 1 Introduction

The Great Recession has highlighted the potential importance of financial factors in business cycle fluctuations. The failures of financial institutions, the increases in funding costs and intermediation spreads, the collapse of the commercial paper and syndicated loans markets—all of these events seem to have led to a significant and prolonged effect on firm output and unemployment. However, the mechanisms by which these events have translated into real economic declines remain in dispute.

In particular, financial macro models that associate output losses to events in financial markets face empirical challenges. First, when constraints are placed on the funding of investment, these models cannot deliver significant output fluctuations.<sup>1</sup> This point is furthermore emphasized by evidence from Chari, Christiano, and Kehoe (2010) who show that in the aggregate, non-financial corporations can finance their capital expenditures entirely from their retained earnings and dividends alone. This suggests that perhaps only a fraction of firms may be constrained in accessing credit; work by Kahle and Stulz (2012) suggests that this may in fact be the case. Viewing this through the lens of a representative firm model leads to the potential conclusion that financial constraints on firms can play only a minor role in business cycle fluctuations.<sup>2</sup>

This paper argues that financial frictions can have much more power in explaining aggregate declines when one takes into account the fact that firms engage in a substantial amount of trade with each another. Consider an economy in which firms produce in isolation but need to obtain funding for their factors of production, i.e. labor and capital. In this case, if a fraction of firms produce less due to some financial shock, this has no direct effect on the other firms in the economy.<sup>3</sup> On the other hand, consider an economy in which firms are interconnected: firms not only purchase labor and rent capital, but also sell and purchase intermediate inputs from one another. All of these transactions are potentially subject to financing constraints. In this interconnected economy, a financial shock to only a subset of constrained firms affects not only these firms but also their trading partners. More specifically, this shock affects the production of the firms that supply intermediate goods to the affected sector, as well as the production of the firms that purchase the affected sector's goods. This in turn affects the firms to which these firms are connected to, and so on and so

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<sup>1</sup>See for example the work of Chari, Kehoe and McGrattan (2007) and Bigio (2009).

<sup>2</sup>This has led to been renewed attention on the affects of financial frictions on the real economy. Many of the theoretical models focus on a representative firm, which when buying continue to buy their inputs (labor, and capital) from the household faces some sort of collateral constraint or financing friction a la Kiyotaki and Moore, Bernanke Gertler (). Another approach is to allow for multiple firms or heterogeneous firms—yet firms

<sup>3</sup>other than through perhaps aggregate effects

forth, resulting in a multiplier effect of the financial shock. Thus, what can be interpreted as small aggregate financial shocks in one sector may affect not only this sector, but propagate to other sectors. The total effect is thus amplified as the financial shock moves down the supply chain.

The key idea in our paper is that when firms are interconnected, working capital must finance not only the primary factors of production such as labor and capital, but all transactions including purchases of intermediate goods. This feature has several general implications for the aggregate effects of financial shocks. First, it implies that the funding of intermediate inputs may require much higher amounts of liquidity than the financing of just labor or capital expenditures.<sup>4</sup> Thus, in stark contrast to a representative firm model, when firms engage in substantial interfirm trade, the aggregate expenditure on the original factors of production (capital and labor) is not sufficient for indicating the bite of financing constraints. Second, a financial shock that may affect only a small portion of interconnected firms can lead to substantial propagation. Thus, aggregate losses due to financial shocks are much larger in models of production networks than those implied by representative firm models. Finally, in a production network, exactly which firms are constrained and how these firms interact with other firms matters. A firm's location within the network will determine how much its own financial state affects the rest of the economy.

**Framework and Results.** In this paper we formalize these ideas by analyzing the distorting effects of financial shocks in economies in which firms are organized into different production network structures. Our goal is to show how production networks matter when firms are subject to financial frictions. We first do this within a very simple example. We consider two basic economies: a horizontal one and a vertical one. In the horizontal economy, firms produce and operate in isolation, their individual products are then aggregated into a final good. In the vertical economy, firms are arranged in a supply chain—each firm buys the good produced by the firm above it and uses it as an input into their production. Finally, in either economy there is a household that supplies labor and consumes the final good. We introduce these two economies in a way such that without any frictions, they are allocatively equivalent: that is, without frictions the equilibrium in either economy yields the exact same allocation, and moreover, this allocation is efficient.

We then ask how allocations change when firms are subject to liquidity constraints. We impose a pledgeability constraint as in Bernanke Gertler (1989), Kiyotaki Moore (1997), or Bernanke, Gertler, Gilchrist (1999): firms each period must pledge a fraction of their

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<sup>4</sup>This point has been informally emphasized by Vincenzo Quadrini in a discussion of Shourideh and Zetlin-Jones (2012).

revenue in order to finance their inputs.<sup>5</sup> This introduces a financial friction such that firms are forced to produce less than their optimal amount. We then compare the equilibrium allocations across the two economies when firms face these constraints.

A summary of our analytic results in this exercise are as follows.

*Result 1: The Labor Wedge.* An aggregate labor wedge arises in either economy between the aggregate marginal rate of transformation of aggregate labor and the marginal rate of substitution. This labor wedge depends on the tightness of the individual firm credit constraints. In the horizontal economy the aggregate labor wedge weighs the credit constraint equally according to each firm's marginal product. On the other hand, in the vertical economy the credit constraint on downstream firms are given greater weight. This implies that constraining downstream has a larger impact on the aggregate labor wedge distortion. This furthermore implies that depending on the network structure, the aggregate labor wedge is an exaggerated measure of the actual financial wedges faced by individual firms.

*Result 2: Aggregate Liquidity.* More vertical transactions imply that more aggregate liquidity in the economy is needed. Specifically, consider any feasible allocation. The amount of liquidity needed in the vertical economy to implement this allocation as an equilibrium is strictly greater than the amount of liquidity needed in the horizontal economy. The intuition for this is quite simple. In the horizontal economy, firms only need to have enough funding to finance their value added. However, in the vertical economy, firms must finance not only their own value added, but also must finance the added value of their intermediate inputs. Thus, there is a double counting of input financing that must occur in the vertical economy. This result is analogous to the quantity theory of money: the more transactions that occur within an economy, the more money-like instruments are needed to finance those transactions.

*Result 3: Network Location of Constraints.* The impact of financial constraints differs by network location. In vertical economies, the most downstream firm has the greatest impact on aggregate output as well as aggregate distortions. That is, a tightening of constraints on the downstream firms leads to a greater decline in output than a tightening of constraints on upstream firms.

*Result 4: Liquidity Multiplier.* A uniform tightening of constraints affects aggregate output differently across the two economies. We call the fall in output due to a uniform fall in liquidity the "liquidity multiplier". The liquidity multiplier in the vertical economy is greater than the liquidity multiplier in the horizontal economy. Thus, tightening constraints in the vertical economy leads to a greater drop in output than in the horizontal economy.

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<sup>5</sup>Underlying this ad-hoc formulation is a more micro-founded explanation: firms require liquid funds to finance its inputs due to a limited enforcement problem. We describe this firm problem in Appendix A.

*Result 5: Spill-over Effects.* There are no direct spill-over effects in the horizontal economy. However, there are direct spill-over effects in the vertical economy. Consider an extension with  $N$  firms arranged in a vertical supply chain and a financial shock hitting firm  $i$ . A tightening of firm  $i$ 's constraint acts like an adverse demand shock on upstream firms (suppliers), while it works as an adverse supply shock on firms downstream (customers).

*Result 6: Measuring Misallocation.* When firms are arranged in production networks with vertical supply chains, the precise measure of misallocation—that is, the one that best indicates the level of distortion in either lower TFP or a greater labor wedge—should be the dispersion in marginal *aggregate* products across firms, not the dispersion in marginal individual products. Dispersion in marginal aggregate products across firms better describes the aggregate distortion in a network economy. Misallocation measured purely by the dispersion in marginal individual products (used, e.g. in Hsieh and Klenow, 2009, and Midrigan and Xu, 2012, among others) is in fact a lower bound for the amount of distortions in the economy.

The results above arise simply from comparing horizontal and vertical economies. However, the world is not organized into simple horizontal and vertical economies. Modern economies feature very intricate and complex networks of production. Thus in order to apply our results to actual economies, we analyze our problem within a general production network structure as in Acemoglu et al. (2012). In this general structure, there are  $N$  sectors and each of these sectors may demand or supply intermediate goods from one another. We solve for the equilibrium in this general economy when firms face financial constraints and then use this to explore the effect of financial shocks in different production networks. In particular, we present results for star, circle, and triangle networks, in addition to the original horizontal and vertical economies.

The main purpose of working with the general network structure is to calibrate the model to data. We use the input-output matrix on the U.S. economy to calibrate the the production network. We use yearly data from 1998 until 2011 on 65 NAICS sectors at the three-digit level, and data from COMPUSTAT on industry costs and sales. We examine the model's implications for output, employment, and the use of inputs. In the calibrated version of our model, we obtain the following results.

*Result 7: The U.S. Liquidity Multiplier.* We compute the U.S. liquidity multiplier which tells us how much aggregate output would drop from a uniform tightening of constraints across all firms. We obtain a multiplier of around 3.8. This means that a 1% tightening in financial constraints should lead to a 3.8% drop in aggregate output.

*Result 8: Implied Drop in Liquidity during the Financial Crisis.* Given the observed drop in

output during the financial crisis, what was the implied fall in liquidity in constrained firms? In our calibrated model using the network structure of the U.S. economy, we find that at the trough of the recession, the required reduction in liquidity would have been around 1.3% of sales for every sector. A model with a representative firm would require at least a 7.6% reduction in liquidity.

*Result 9: Which sectors are most sensitive to liquidity shocks?* Within our calibrated model, we ask: which sectors' output responds most to a decrease in 1% in the liquidity of that sector? We find that manufacturing sectors such as Motors (automotive), Chemicals, Metals, are the most sensitive to own liquidity shocks.

*Result 10: Which sectors have the most Influence?* Within our calibrated model, we ask: which sectors have the greatest effect on aggregate output when their own liquidity drops? We find that the Retail sector has the most influence.

Thus, we believe our paper makes a step in understanding both the theoretical and quantitative implications of production networks in models of financial frictions.

**Layout.** The remainder of this section discusses the related literature while rest of the paper is organized as follows. Section 2 introduces the simple model and characterizes the general equilibrium. Section 3 then explores the implications of networks and financial frictions for business cycles. Here we present the theoretical results outlined above. Section 4 examines the general case of an arbitrary N-by-N production network then explores the effects of financial shocks under different network structures. Finally, in Section 5 we calibrate this model to the input-output structure of the U.S. economy and compute the production network multiplier. Section 6 concludes.

**Related Literature.** Our paper analyzes the impact of financial frictions within different production networks. Thus, the two main strands of literature to which our paper fits is (i) the literature on financial frictions, and (ii) the literature on production networks. First, there is a large and growing literature on the real effects of firm financial constraints, starting with the seminal work of Bernanke Gertler (1989), Kiyotaki Moore (1997), and Bernanke, Gertler, Gilchrist (1999). This earlier literature has focused on financial accelerators in representative firm environments. A more recent literature has looked at different forms of firm heterogeneity: entrepreneurial ability and net worth (Buera), adverse selection (Bigio 2012, Kurlat 2012), dispersed information (La'O, 2010), among others. Buera and Moll (2012) in fact study three variants: heterogeneity in final good productivity, heterogeneity in investment costs, and heterogeneity in recruiting costs of labor. In contrast to all of this previous work, ours is the first paper, to our knowledge, to highlight the role of the

organization of production structures or networks in amplifying financial shocks. Thus, even without firm heterogeneity, the underlying organizational structure of the economy becomes a key determinant for how important financial frictions are for aggregate fluctuations.

The second strand of literature our paper falls under is that of production networks. In this sense, our paper shares the spirit of the early literature on real business cycles and the role of sectoral shocks. We build on Long and Plosser’s (1983) multi-sectoral model of real business cycles. Following Long and Plosser, a debate ensued between Horvath (1998, 2000) and Dupor (1999) over whether sectoral shocks could lead to strong observable aggregate TFP shocks. More recently, this work has been extended and generalized by Acemoglu et al. (2011), for arbitrary production networks, thereby providing a general mathematical framework to answer these questions. The results of the Acemoglu et al. paper are related to that of Gabaix (2011), who showed that firm level shocks may translate into aggregate fluctuations when the firm size distribution is power law distributed, i.e. sufficiently heavy-tailed. We use the Acemoglu et al. (2011) paper as a basis for our model in Section 4.<sup>67</sup>

Our paper is most related to that of Jones (2009 and 2011). In these papers, Jones shows that in economies with intermediate goods there arises a multiplier effect on productivity. In Jones (2009), he considers a particular network in which all firms purchase a uniform intermediate good (which is itself composed of all produced goods); in this, all firms are equally important in terms of the network. In Jones (2011) on the other hand, he considers a more general network, similar to that in Section 4 of our paper, and applies this framework by considering different input-output structures of various economies. He then computes implied productivity multipliers, and uses this to explain cross-country differences in long-run growth.

We think our paper very much shares the same spirit as Jones (2009 and 2011)—we too want to highlight how trade in intermediate goods leads to amplification. However, although our framework is similar, our paper differs from Jones (2009 and 2011) in two important respects. First, we focus more on the idea that firm-level distortions are caused by financing frictions. Under this interpretation, we can use our model to understand the role of drops of aggregate liquidity and its multiplier effect on output (as opposed to the multiplier on TFP). We can also use our model to answer the question of how the network structure itself affects the amount of liquidity needed in the economy. Furthermore, our framework is then suitable for understanding short run phenomena such as the recent financial crisis, which we analyze

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<sup>6</sup>Relatedly, Cooper and Haltiwanger (1990), Jovanovic (1987), and Durlauf (1993), have explored setups in which strong complementarities across firms can generate aggregate fluctuations from micro shocks.

<sup>7</sup>Finally, most papers including ours takes the production network of the economy as exogenous, while a recent paper by Oberfield (2011) provides a theory of the formation and evolution of input-output structures.

in Section 5. This gives us an advantage in terms of identifying distortions in the data. In particular, Jones (2011) writes, “there is a fundamental identification problem: we see data on observed intermediate goods shares, and we do not know how to decompose that data into distortions and differences in technologies. This identification problem is not solved in anything I have done” Our approach gives us some guidance in dealing with this identification issue. Under the assumption that the recent crisis was caused by tightening of financial constraints, we can do a simple exercise in which we assume that pre-crisis, differences in intermediate good shares were due primarily to technology differences. However, any changes in shares during the crisis arose primarily from financial friction distortions, rather than movements in sector-specific shares. This allows us to then calibrate technology shares and financial distortions. Although ours is not a perfect strategy, we can then make some progress in obtaining a number for the liquidity multiplier.

Second, while Jones (2009) and Jones (2011) were more focused on aggregate effects of input-output structures, we in addition focus our attention on individual sectors, and the differential impact of these sectors on aggregate variables due to their network *location*. In the language of Acemoglu et. al (2011), we focus on properties of the influence vector which summarizes how much each firm or sector matters. In the simple model this allows us to study the effect firm financial constraints have on aggregate output and aggregate distortions—we find that in vertical economies, the downstream firms have the most impact. Furthermore, when we move to the calibrated version, our general network then helps us to identify particular sectors of the U.S. economy which are the most important in terms of aggregate output declines.

Our paper also provides implications for the literature on wedges and misallocation, see e.g. Chari, Kehoe, McGrattan (2007), Hsieh and Klenow (2009), Lagos (2006), Restuccia and Rogerson (2008), Banerjee and Duflo (), Midrigan and Xu (2012). This literature has emphasized that distortions at the micro or sectoral level can lead to aggregate distortions or wedges that we observe in aggregate data. In relation to this literature, our model produces two important implications. First, we obtain an aggregate labor wedge which is a weighted combination of sectoral labor wedges. However, we find that the network structure determines the weight on each individual wedge in the aggregate wedge—more distortions on downstream firms have more weight than distortions affecting upstream firms. Second, many of the empirical papers in this literature have simply measured misallocation as the dispersion in marginal products. While this would be a correct measure in horizontal economies for indicating aggregate TFP or labor wedge distortions,<sup>8</sup> this is an imprecise measure in more

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<sup>8</sup>Or economies in which all firms have equal influence in the network. See e.g. Jones (2009)



general network economies. We contend that a more precise measure should be the Marginal Aggregate product, which it takes into account the influence of each firm on aggregate TFP and wedges due to the network structure.

Our paper also highlights the importance of vertical supply chains. There has been other work in this area. Bak et al. (1993) stresses the importance of supply chains in aggregate fluctuations. Kiyotaki and Moore (1997) study what they call credit chains. In their setting firms borrow from each other, and a temporary shock to the liquidity of some firms causes a chain reaction in which other firms also get in financial difficulty. Their model differs from ours because of the timing of payments is an important component, whereas our model is static. More recently, Kim and Shin (2011) study an environment in which firms face recursive moral hazard in a supply chain resulting in interlocking claims and obligations. These financial linkages serve as the “glue” that ties firms together in the supply chain. Levine (2010) builds a theory of production chains in which chains are fragile because they are subject to the weakest link, as in Kremer’s (1993) O-ring story. Optimal chain length in Levine’s model is determined by the trade-off between the gains to specialization and the higher failure rate associated with longer chain length.

Furthermore, our paper makes a step in trying to quantify the empirical effects of networks. A recent paper by Di Giovanni, Levchenko, Mejean (2012) provide empirical evidence on how firm-specific shocks generate to aggregate fluctuations. They test whether this can be accounted for by the granularity effect as in Gabaix (2010) or the the linkages effect as in Acemoglu et. al (2012). They find that firm linkages are about twice as important as granularity. Furthermore, Raddatz (2010) provides some evidence for the credit chains mechanism. Chodorow-Reich (2013) provides evidence on constraints affecting the use of variable inputs.

Finally, at a very basic level our paper is related to Friedman’s quantity theory of money, which states that  $MV = PT$ , where  $M$  is money,  $V$  is velocity,  $P$  is the aggregate price level and  $T$  is the real value of aggregate transactions. First, hold velocity constant and normalize the aggregate price level to 1. Money in our economy can be thought of as anything that can be used to purchase goods—in our model we call this liquidity. Thus, we show that more vertical networks implies a greater number or value of transactions. More transactions implies more liquidity is needed, similarly as in the quantity theory. This is the main idea behind our second result: more vertical economies imply more transactions, which implies more liquidity is needed to implement any particular allocation.

## 2 Two Simple Economies

In this section we consider a very simple example which illustrates the main idea of how the production network structure interacts with the financial frictions. We consider the implications of collateral constraints in two economies which differ in their organizational structure of production. The vertical economy is an economy in which firms are arranged in a vertical supply chain. The horizontal economy is then engineered to be allocatively equivalent. We then characterize the general equilibrium in both economies

*Vertical Economy.* There are three firms that use labor and intermediate inputs to produce output. We assume that these firms are perfectly competitive, in that they take prices as given.<sup>9</sup> The firms are organized in a vertical supply chain. Specifically, their production functions are given by

$$\begin{aligned}y_{v1} &= A_1 n_{v1}^{\alpha_1} \\y_{v2} &= A_2 n_{v2}^{\alpha_2} y_{v1}^{\beta_2} \\y_{v3} &= A_3 n_{v3}^{\alpha_3} y_{v2}^{\beta_3}\end{aligned}$$

where  $y_{vi}$  is the amount produced by firm  $i$  and  $n_{vi}$  is the amount of labor employed by firm  $i$ . Thus, firm 1 uses labor as its sole input, however for  $i = 2, 3$ , firm  $i$  also uses as input the output of firm  $i - 1$ . Finally, the final consumption good is the output of firm three, that is  $Y_v = y_{v3}$ . We can therefore write the aggregate production function of the economy in terms of labor as follows

$$Y_v = y_{v3} = A_3 n_{v3}^{\alpha_3} (A_2 n_{v2}^{\alpha_2})^{\beta_3} (A_1 n_{v1}^{\alpha_1})^{\beta_2 \beta_3}$$

For simplicity, assume CRS:  $\alpha_3 + \alpha_2 \beta_3 + \alpha_1 \beta_2 \beta_3 = 1$ . Figure 2 illustrates the flow of inputs and output in the vertical economy.

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<sup>9</sup>One can think of this simply as three sectors, each composed of a continuum of perfectly competitive firms.

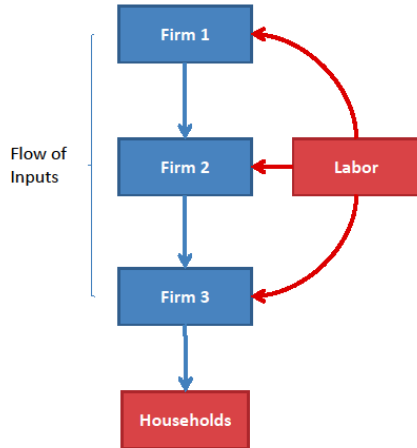


Figure 1: Vertical Economy

*Horizontal Economy.* Now consider an equivalent, but completely horizontal economy. There are three representative firms that use labor to produce a good. The production functions of these representative firms are as follows

$$\begin{aligned}
 y_{h1} &= A_1 n_{h1}^{\alpha_1} \\
 y_{h2} &= A_2 n_{h2}^{\alpha_2} \\
 y_{h3} &= A_3 n_{h3}^{\alpha_3}
 \end{aligned}$$

where  $y_{hi}$  is the amount produced by firm  $i$  and  $n_{hi}$  is the amount of labor employed by firm  $i$ . These three goods aggregated into a final consumption good,  $Y_h$ . We normalize this consumption basket so as to make it equivalent to the final good in the vertical economy:

$$Y_h = Y_v$$

$$Y_h = y_{h1}^{\beta_2 \beta_3} y_{h2}^{\beta_3} y_{h3}$$

Therefore, both economies have the same aggregate production function. However, unlike the vertical economy, the firms in the horizontal economy operate in isolation from one another, only combining at the end in terms of consumption, as in Dixit-Stiglitz. Figure 2 illustrates the flow of inputs and output in the horizontal economy.

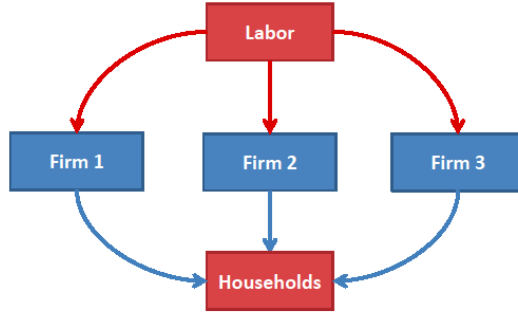


Figure 2: Horizontal Economy

*Households and Market Clearing.* To close the economy we introduce households. In either economy, there is a representative household with preferences given by

$$U(C) - V(N)$$

where  $U : \mathbb{R} \rightarrow \mathbb{R}$  is increasing and concave,  $V : \mathbb{R} \rightarrow \mathbb{R}$  is increasing and convex,  $C$  is the final good consumption, and  $N$  is labor supplied competitively to the market. The household's budget constraint is given by

$$C = wN$$

where  $w$  is the competitive real wage rate and where we have normalized the price of the final good to 1. Finally, for markets to clear, we have that consumption is equal to output  $Y = C$ , and labor supply equals labor demand  $N = n_1 + n_2 + n_3$ .

*Remarks and Notation.* We will use  $\varepsilon \in \{v, h\}$  to denote the economy of interest, where  $\varepsilon = v$  denotes the vertical economy and  $\varepsilon = h$  the horizontal.

Our first remark is that in this paper we abstract from investment. The model is static, so that firms only have static inputs. As one will see later, the financial constraint will be on working capital.

Second, note that we are taking the network structure of these economies as exogenous. As will be seen later, there will be incentives for firms to merge or vertically integrate when there are frictions. This then begs the question of why firms are structured in the economy as they are. The theory of the firm and its boundaries is an interesting one, however, here

we abstract from these considerations and take the firm boundaries as given. See Antras (2011) for a sequential production chain regarding the optimal allocation of ownership rights along a supply chain.

*Equilibrium Definition.* We define the competitive equilibrium in either economy as follows

**Definition 1.** A competitive equilibrium in economy  $\varepsilon \in \{v, h\}$  is a collection of quantities  $\{n_{\varepsilon 1}, n_{\varepsilon 2}, n_{\varepsilon 3}, y_{\varepsilon 1}, y_{\varepsilon 2}, y_{\varepsilon 3}, N_{\varepsilon}, Y_{\varepsilon}\}$  and prices  $\{p_{\varepsilon 1}, p_{\varepsilon 2}, p_{\varepsilon 3}, w_{\varepsilon}\}$  such that

- (i) each representative firm maximizes profits,
- (ii) the representative household maximizes utility,
- (iii) markets clear.

This is a standard definition for a Walrasian equilibrium in a production economy.

## 2.1 Frictionless Benchmark

As a benchmark, we first consider the equilibrium in either economy in the absence of frictions. By construction, the vertical and horizontal economies are allocatively equivalent.

**Proposition 1.** In either economy without frictions, there exists a unique equilibrium allocation. In either economy  $\varepsilon \in \{v, h\}$ , the unique equilibrium allocation is given by

$$\tilde{\alpha}_3 \frac{Y_{\varepsilon}}{n_{\varepsilon 3}} = \tilde{\alpha}_2 \frac{Y_{\varepsilon}}{n_{\varepsilon 2}} = \tilde{\alpha}_1 \frac{Y_{\varepsilon}}{n_{\varepsilon 1}} = \frac{V'(N)}{U'(Y)} \quad (1)$$

and

$$N = n_1 + n_2 + n_3$$

where

$$\tilde{\alpha}_3 = \alpha_3, \quad \tilde{\alpha}_2 = \alpha_2 \beta_3, \quad \tilde{\alpha}_1 = \alpha_1 \beta_2 \beta_3$$

denote the each firm's labor share in the aggregate production function.

The aggregate production function simply transforms each type of labor into aggregate output. In equilibrium, the marginal rate of transformation of each type of labor is equal to the marginal rate of substitution. Thus, absent any frictions, the way production of an economy is broken down into different firms or network structures is irrelevant. This is a simple extension of the Modigliani-Miller () result. Here, instead of considering how an individual firm is sliced up in terms of financing, we consider how a macroeconomic production function is sliced up into different units. Aside from any frictions, for a given

aggregate production function, the way it is cut into different production units does not matter for allocations.

Proposition 1 further implies that we may write

$$\bar{\alpha} \frac{Y}{N} = \frac{V'(N)}{U'(Y)}$$

where  $\bar{\alpha} = \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3$  is the total labor share of output. Thus, these economies admit a representative firm, with production function  $Y = \bar{A}N^{\bar{\alpha}}$ , where  $\bar{A} = A_1^{\beta_2\beta_3}A_2^{\beta_3}A_3$  is aggregate productivity.

### 3 Implications of Financial Frictions

After establishing the allocational equivalence of the two economies in the absence of frictions, we now consider the implications of adding financial frictions. Financial frictions introduce distortions into the economy, relative to the frictionless benchmark; however, depending on the network structure, these frictions distort the two economies in different ways.

We introduce financial frictions by adding collateral constraints on input purchases. We assume that each firm faces a constraint in which their expenditure on inputs is constrained to be less than or equal to a fraction  $\chi$  of their revenue. One can think of this as follows: firms can credibly commit to pay only a fraction  $\chi$  of their revenue to laborers or suppliers, and can abscond with the rest after production and sales are realized. Hence, expenditure on inputs cannot exceed the pledgeable portion of their revenue. In the horizontal economy this pledgeability constraint is given by the following.

$$w_h n_{hi} \leq \chi_{hi} p_{hi} y_{hi} \tag{2}$$

On the other hand, in the vertical economy, only firm 1 uses labor as a sole input so faces a constraint as in (2), while firms 2 and 3 face the following constraint

$$w_v n_{vi} + p_{v,i-1} y_{v,i-1} \leq \chi_{vi} p_{vi} y_{vi} \tag{3}$$

Thus, firms face working capital constraints.<sup>10</sup>

The financial frictions introduce distortions into the two economies. However, we note that these constraints are not directly comparable across the two economies, as the vertical

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<sup>10</sup>This working-capital constraint is similar to the static-input financing example in Chari, Kehoe, and McGrattan ().

economy firms must finance both labor and intermediate goods whereas firms in the horizontal economy need only finance their wage bill. We take this into consideration in our analysis. That is, our results will not depend on how  $\chi_{hi}$  directly compares to  $\chi_{vi}$ , but instead will rest on how these constraints manifest themselves differently in terms of distorting the two economies.

*Equilibrium Characterization.* We first examine these distortions at the individual firm level. The pledgeability constraint faced by any firm introduces a wedge between the firm's marginal benefit and marginal cost of production. In the horizontal economy, each firm's production is pinned down by

$$w_h = \phi_{hi} p_{hi} \alpha_i \frac{y_{hi}}{n_{hi}}$$

where

$$\phi_{h1} = \min \left\{ 1, \frac{\chi_{h1}}{\alpha_1} \right\}, \quad \phi_{h2} = \min \left\{ 1, \frac{\chi_{h2}}{\alpha_2} \right\}, \quad \phi_{h3} = \min \left\{ 1, \frac{\chi_{h3}}{\alpha_3} \right\} \quad (4)$$

This simply states that the marginal cost is equal to the marginal benefit, times some wedge. This individual wedge represents the distortion for that firm away from its optimal labor usage due to the collateral constraint. That is, for any firm  $i$ , the wedge  $\phi_{hi} \in [0, 1]$ . When  $\chi_{hi} < \alpha_i$ , the firm's pledgeability constraint is binding, and the wedge is given by  $\phi_{hi} = \chi_{hi}/\alpha_i$ . The constraint is binding whenever  $\chi_{hi}$  is less than the labor share of output of firm  $i$ . On the other hand, if  $\chi_{hi} \geq \alpha_i$ , then the firm's pledgeability constraint is not binding—firms operate at their optimal level and have enough funds to cover their expenses. In this case, there is no wedge between the firm's marginal benefit and marginal cost of production, i.e.  $\phi_{hi} = 1$ .

In the vertical economy, firms solve a cost minimization problem in terms of its expenditure on each of its inputs: labor and the intermediate good. This cost minimization implies that their expenditure on each good is equal to the ratio of the relative shares of each input in production.

$$\frac{w_v n_{vi}}{p_{v,i-1} y_{v,i-1}} = \phi_{vi} \alpha_i$$

Given this condition, each firm's production is then pinned down by the following condition

$$w_v = \phi_{vi} p_{vi} \alpha_i \frac{y_{vi}}{n_{vi}}$$

where

$$\phi_{v1} = \min \left\{ 1, \frac{\chi_{v1}}{\alpha_1} \right\}, \quad \phi_{v2} = \min \left\{ 1, \frac{\chi_{v2}}{\alpha_2 + \beta_2} \right\}, \quad \phi_{v3} = \min \left\{ 1, \frac{\chi_{v3}}{\alpha_3 + \beta_3} \right\} \quad (5)$$

Again, for any firm  $i$ , the wedge  $\phi_{hi} \in [0, 1]$  represents the distortion in optimal production level due to the collateral constraint. This the same condition as in the horizontal economy; however, the only difference here is that for firms 2 and 3, the constraint is binding whenever  $\chi_{vi} < \alpha_i + \beta_i$ , that is, when the pledgeability ratio  $\chi$  is less than the total output share of inputs (the labor share plus the share of intermediate goods).

Combining the individual firm conditions with market clearing and household optimality conditions, we reach the following proposition which fully characterizes the equilibrium allocation.

**Proposition 2.** *Suppose firms face pledgeability constraints*

(i) *In the horizontal economy, the unique equilibrium allocation is given by*

$$\begin{aligned} (\phi_{h3}) \tilde{\alpha}_3 \frac{Y_h}{n_{h3}} &= V'(N_h) / U'(Y_h) \\ (\phi_{h2}) \tilde{\alpha}_2 \frac{Y_h}{n_{h2}} &= V'(N_h) / U'(Y_h) \\ (\phi_{h1}) \tilde{\alpha}_1 \frac{Y_h}{n_{h1}} &= V'(N_h) / U'(Y_h) \\ N_h &= n_{h1} + n_{h2} + n_{h3} \end{aligned}$$

(ii) *In the vertical economy, the unique equilibrium allocation is given by*

$$\begin{aligned} (\phi_{v3}) \tilde{\alpha}_3 \frac{Y_v}{n_{v3}} &= V'(N_v) / U'(Y_v) \\ (\phi_{v2} \phi_{v3}) \tilde{\alpha}_2 \frac{Y_v}{n_{v2}} &= V'(N_v) / U'(Y_v) \\ (\phi_{v1} \phi_{v2} \phi_{v3}) \tilde{\alpha}_1 \frac{Y_v}{n_{v1}} &= V'(N_v) / U'(Y_v) \\ N_v &= n_{v1} + n_{v2} + n_{v3} \end{aligned}$$

In either economy, for each type of labor there is now a wedge between its marginal rate of transformation into aggregate output and its marginal rate of substitution. In the horizontal economy this wedge between the MRT and the real wage is simply the same wedge that arises in the individual firm decisions. This is due to the fact that each firm in the horizontal economy operates in isolation. Whatever distortion shows up at the firm level only effects the marginal rate of transformation for that type of labor, but not for others. On the other hand, in the vertical economy, this is not the case. The individual wedge of firm 2 affects the wedge for both firm 2 and firm 1. Similarly, the wedges of firm 3 affect the wedges found in firms 1 and 2. Thus, the downstream financial frictions distort upstream input use. As in Jones (2011), it has a multiplier effect, which we will evaluate in the next subsection.



Finally, note that these economies with financial frictions are isomorphic to an economy without frictions, but with taxes given by  $(1 - \tau_i) = \phi_i$  for  $\tau \in [0, 1]$ . That is, it is isomorphic to an economy where firms face taxes but not subsidies. This relates to the literature on taxation and supply chains, for example, the input-output model of Jones (2011).

### 3.1 The Aggregate Labor Wedge

We now look at how these distortions at the individual firm level affect the economy at the aggregate level. One variable of interest is the aggregate labor wedge. A large literature has documented large labor wedges as well as countercyclicality of this wedge (e.g., Hall, 1997; Rotemberg and Woodford, 1999; Chari, Kehoe, and McGrattan, 2007; Shimer, 2009). Following the literature, we define the aggregate labor wedge  $(1 - \tau)$  implicitly by

$$(1 - \tau) \bar{\alpha} \frac{Y}{N} = \frac{V'(N)}{U'(C)}$$

That is, the aggregate labor wedge is simply the wedge between the aggregate marginal rate of transformation of aggregate labor, and the marginal rate of substitution. Using our results for the equilibrium, we may also back out the aggregate labor wedge in these economies. Note that in the frictionless economy,  $\tau = 0$  so that there is no wedge or aggregate distortion. The following proposition then characterizes the aggregate labor wedge when firms face pledgeability constraints.

**Proposition 3.** *Suppose firms face pledgeability constraints. (i) In the horizontal economy, the aggregate labor wedge is given by*

$$(1 - \tau_h) = \tilde{\alpha}_3 (\phi_{h3}) + \tilde{\alpha}_2 (\phi_{h2}) + \tilde{\alpha}_1 (\phi_{h1})$$

*(ii) In the vertical economy, the aggregate labor wedge is given by*

$$(1 - \tau_v) = \tilde{\alpha}_3 (\phi_{v3}) + \tilde{\alpha}_2 (\phi_{v2}\phi_{v3}) + \tilde{\alpha}_1 (\phi_{v1}\phi_{v2}\phi_{v3})$$

Thus, the pledgeability constraints introduce aggregate labor wedges between the aggregate marginal product of labor and the real wage. Proposition 3 makes clear that the aggregate labor wedge is simply a weighted sum of the individual labor wedges—the weighting is simply given by each labor share of aggregate output. However, depending on how the constraints  $\phi$ 's affect the individual labor wedges, the aggregate labor wedges differ across the two economies. In the horizontal economy all constraints are weighted according to

their respective labor share. On the other hand, in the vertical economy, the downstream constraint,  $\phi_3$ , has the greatest impact on the aggregate wedge. This is because downstream constraints distorts upstream labor choices.

Thus, we see that the way the economy is sliced into different sectors and components, and where each constraint binds, matters for the aggregate labor wedge. That is, when there are financial frictions, different network structures lead to different aggregate distortions.

Implications for the interest rate–aggregate labor wedge is the sum of individual wedges. so individual wedges could be incredibly small.

A number of other papers have shown that financial frictions lead to static wedges. Chari, Kehoe, McGrattan (2007) consider a static-input financing friction and show that this leads to an efficiency wedge.<sup>11</sup> Buera and Moll (2012) also find a labor wedge due to financial frictions in a search model of labor. Here, we show that if individual firms face constraints, the network structure is important in determining how each constraint contributes to the aggregate labor wedge. This result shows that financial frictions can lead to labor wedges; in particular, this labor wedge will depend on the network structure.

### 3.2 Aggregate Liquidity

Another way these two economies differ is in the aggregate amount of liquidity needed for any allocation. Let us first define the aggregate amount of pledgeable funds, i.e. liquidity, as follows.

**Definition 2.** For economy  $\varepsilon \in \{v, h\}$ , let  $M_\varepsilon$  denote the aggregate amount of liquidity

$$M_\varepsilon \equiv \chi_{\varepsilon 1} p_{\varepsilon 1} y_{\varepsilon 1} + \chi_{\varepsilon 2} p_{\varepsilon 2} y_{\varepsilon 2} + \chi_{\varepsilon 3} p_{\varepsilon 3} y_{\varepsilon 3}$$

That is, we define liquidity  $M_\varepsilon$  to be the aggregate amount of pledgeable funds. As we've mentioned previously, we cannot directly compare the constraints across the two economies. Hence, fixing the vectors  $\{\}$  and  $\{\}$  and then comparing the liquidity across the two economies would be uninformative. However, we can instead ask what is the liquidity needed in either economy to implement a given allocation. First, we define a feasible allocation in the economy as follows.

**Definition 3.** An allocation  $\{n_1, n_2, n_3, N, Y\}$  is feasible if and only if  $n_1 + n_2 + n_3 = N$  and  $Y = A_3 n_{v3}^{\alpha_3} (A_2 n_{v2}^{\alpha_2})^{\beta_3} (A_1 n_{v1}^{\alpha_1})^{\beta_2 \beta_3}$ .

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<sup>11</sup>See Section ?? of their paper.

Thus, an allocation is feasible if and only if it satisfies the economy's resource constraints. Now, suppose we fix a feasible allocation. In order to implement this allocation as an equilibrium outcome, what is the minimum amount of liquidity needed in order to do so?

**Proposition 4.** *Fix some feasible allocation  $\{n_1, n_2, n_3, N, Y\}$ . Then, the minimum liquidity needed to implement this allocation is given by*

(i) *the liquidity needed in the horizontal economy to achieve this allocation is given by*

$$M_h = \frac{V'(N)}{U'(Y)}N \quad (6)$$

(ii) *the liquidity needed in the vertical economy to achieve this allocation is given by*

$$M_v = \frac{V'(N)}{U'(Y)} \left( N + \frac{\beta_2}{\alpha_2}n_2 + \frac{\beta_3}{\alpha_3}n_3 \right) \quad (7)$$

Therefore

$$M_v > M_h$$

Thus, we find that the amount of liquidity needed to implement any feasible allocation is strictly greater in the vertical economy than in the horizontal economy. Note that this proposition is stated in terms of the allocation alone, not in terms of the constraints  $\phi$ , thereby making the two measures directly comparable.

The intuition for this result is quite simple. In the horizontal economy, firms need only to finance their own cost of labor, that is, their own added value. Thus, the aggregate amount of liquidity needed to implement a feasible allocation is simply just the sum of the equilibrium wage bills

$$M_h = wn_1 + wn_2 + wn_3$$

where the real wage  $w$  is equal to the marginal rate of substitution  $V'(N)/U'(C)$  in equilibrium. In the vertical economy, on the other hand, it is as if there is double counting. In addition to the value of their own labor, firms in the vertical economy must also finance their expenditure on intermediate goods, thereby also pledging collateral for the labor purchased upstream.

$$M_v = wn_1 + \left( wn_2 + \frac{1}{\chi_1}wn_1 \right) + \left( wn_3 + \frac{1}{\chi_2}wn_2 + \frac{1}{\chi_2\chi_1}wn_1 \right)$$

Furthermore, from comparing (6) to (7), we see that the greater the output shares  $\beta_2$  and  $\beta_3$  of the intermediate goods, the greater the amount of liquidity needed in the vertical economy relative to the horizontal in order to implement the same allocation.

Moreover, note that in the vertical economy the aggregate amount of liquidity is greater than aggregate expenditure on labor; yet, despite this difference in liquidity and labor expenditure, collateral constraints are still binding. Chari, Christiano, Kehoe (2009) find that in the aggregate, among public companies, retained earnings plus dividends are greater than capital expenditures. One potential conclusion that they draw from this is that financial frictions do not matter, as firms could clearly finance their own capital expenditures with their own liquidity. There are a number of caveats to this finding—first, that these are only public companies; second, that this is only looking at the aggregate rather than individual firms and hence not taking into account distributional effects. Our findings here challenge this conclusion in another way—which firms are constrained and where they are in the production network matter. Here, in the vertical supply chain, the aggregate amount of funds in equilibrium is greater than the aggregate expenditure on labor (not including intermediate inputs), yet firms are still constrained by their collateral. If instead this were a representative firm economy or a horizontal economy, this would not be the case. Thus, the conclusion we obtain from this simple exercise is that the aggregate amount of available funds may not indicate the bite of financial frictions.

Finally, we see a close relationship here to the quantity theory of money. Effectively, one can think of the pledgeability constraints on the firms as analogous to cash-in-advance constraints. Money can be thought of as anything used to make transactions. Hence, the aggregate amount of pledgeable funds is similar to the amount of money in the economy. With this interpretation in mind, our results are similar to the following representation of the quantity theory of money,

$$MV = PT$$

where  $M$  is money,  $V$  is the velocity of money,  $P$  is the aggregate price level, and  $T$  is the aggregate number of transactions. Suppose that the aggregate price level  $P$  is normalized to 1 and that the velocity  $V$  is a constant. This implies that the level of transactions in the economy is proportional to amount of money. This general concept holds true in our model: more money is needed in the vertical economy than in the horizontal economy because there is a greater level of transactions occurring between firms. Therefore, given any equilibrium allocation, the more transactions made in the economy, that is, the more times goods change hands between firms, the more money is necessary to complete these transactions and implement the allocation.

### 3.3 Aggregate effects of Tightening Constraints

Next, we consider how the tightening of these collateral constraints affects aggregate output. For simplicity, we specify a particular utility function in order to solve for aggregate output in closed form. Suppose that utility over consumption and labor is given by

$$U(C) - V(N) = \log C - N$$

The assumption of log-linear utility over consumption and linear disutility of labor is not crucial for any of our results, but simplifies the expressions considerably. Given this specification, aggregate output is given in the following Lemma.

**Lemma 1.** (i) *Aggregate output in the frictionless economy is given by*

$$\bar{Y} = \bar{A} (\tilde{\alpha}_3)^{\tilde{\alpha}_3} (\tilde{\alpha}_2)^{\tilde{\alpha}_2} (\tilde{\alpha}_1)^{\tilde{\alpha}_1}$$

(ii) *Suppose firms face pledgeability constraints and fix vectors  $\{\phi_{v1}, \phi_{v2}, \phi_{v3}\}$  and  $\{\phi_{h1}, \phi_{h2}, \phi_{h3}\}$ . Given these constraints, aggregate output in the horizontal economy is given by*

$$Y_h = \bar{Y} (\phi_{h3})^{\tilde{\alpha}_3} (\phi_{h2})^{\tilde{\alpha}_2} (\phi_{h1})^{\tilde{\alpha}_1} \quad (8)$$

*and aggregate output in the vertical economy is given by*

$$Y_v = \bar{Y} (\phi_{v3})^{\tilde{\alpha}_3} (\phi_{v2}\phi_{v3})^{\tilde{\alpha}_2} (\phi_{v1}\phi_{v2}\phi_{v3})^{\tilde{\alpha}_1} \quad (9)$$

Lemma 1, thus provides expressions for aggregate output in the frictionless economy, as well as in the vertical and horizontal economies with financial frictions. Note that because  $\phi \in [0, 1]$  and strictly less than 1 whenever collateral constraints are binding, this implies that output in the horizontal economy is lower when constraints are binding than when they are not, as expected. Similarly output in the vertical economy is lower when constraints are binding than when they are not. Using these expressions for aggregate output, we now consider the aggregate effect of tightening individual collateral constraints. We obtain the following result.

**Proposition 5.** *In the horizontal economy*

$$\begin{aligned}\frac{d \log Y_h}{d \log \phi_{h1}} &= \tilde{\alpha}_1 > 0 \\ \frac{d \log Y_h}{d \log \phi_{h2}} &= \tilde{\alpha}_2 > 0 \\ \frac{d \log Y_h}{d \log \phi_{h3}} &= \tilde{\alpha}_3 > 0\end{aligned}$$

*In the vertical economy*

$$\begin{aligned}\frac{d \log Y_v}{d \log \phi_{v1}} &= \tilde{\alpha}_1 \\ \frac{d \log Y_v}{d \log \phi_{v2}} &= \tilde{\alpha}_2 + \tilde{\alpha}_1 \\ \frac{d \log Y_v}{d \log \phi_{v3}} &= \tilde{\alpha}_3 + \tilde{\alpha}_2 + \tilde{\alpha}_1\end{aligned}$$

Consider first the horizontal economy. The effect on aggregate output of tightening any of the collateral constraints is simply equal to its labor share of total output. That is, tightening a collateral constraint leads to a 1-for-1 decrease in the labor employed at firm  $i$ . This is due to the fact that with log-utility over consumption and linear disutility of labor, income and substitution effects cancel, and thus labor simply falls 1-for-1 with the collateral constraint. The effect of a fall in labor also affects aggregate output according to its labor share.

On the other hand, in the vertical economy the constraints downstream have a greater impact on aggregate output than those upstream. For example, a percentage change in the collateral constraint of firm 1 leads to the a fall in aggregate output equal to its labor share—the same as in the horizontal case. In contrast, a percentage change in the collateral constraint on firm 3 has a greater effect than its own labor share, instead it is the sum of the labor shares of all firms 1, 2, and 3. The reason for this is that not only is there a direct effect on firm 3 employment, but it also directly affects the labor chosen by firms 1 and 2, due to reduced demand for their products. We study these spill-over effects more closely in the following subsection. For now, we see that in terms of aggregate output, downstream liquidity has the largest effect.

Finally, suppose all constraints were to tighten at the same time—how much would aggregate output fall in response to this aggregate tightening?

**Proposition 6.** *Suppose we scaled down all collateral constraints by  $x$  percent so that each firm faces a collateral constraint given by  $\phi(1 - x)$ . Then aggregate output falls more in the*

vertical economy than in the horizontal economy

$$\frac{d \log Y_v}{d \log x} < \frac{d \log Y_h}{d \log x} < 0$$

That is, the liquidity multiplier is greater in the vertical economy than in the horizontal.

Formally, suppose we scaled down all constraints by  $x$  percent. The drop in aggregate output due to this fall in aggregate liquidity is given by  $d \log Y / d \log x < 0$ . We call this object  $|d \log Y_v / d \log x|$  the liquidity multiplier—it tells us how much aggregate output falls due to a 1 percent decrease in collateral constraints across the board. First, note that in the horizontal economy, this multiplier is equal to the aggregate labor share, and therefore Note that if the share of labor is 1, as in CRS production function, then the liquidity multiplier must be equal to 1. On the other hand, the multiplier in the vertical economy must necessarily be greater than 1. In fact, in our calibration results in Section 5, we find a liquidity multiplier of 3.5 in the U.S. economy. In conclusion, aggregate output falls more in the vertical economy than in the horizontal economy.

### 3.4 Spill-over effects of Tightening Constraints

In the previous subsection we explored the aggregate effects of tightening credit constraints. However, underlying these results is the behavior of individual firms and prices in response to any liquidity shock. To understand this, we now look at the spill-over effects from tightening the collateral constraints of an individual firm. How does the tightening of a constraint of one firm affect the production of other firms in the economy? First, for the horizontal economy, clearly there are no direct spill over effects since there are no linkages among firms. The only effects that could be coming from the centralized labor market—income effects on the wage. Our specification for utility kills these indirect effects (see appendix for an explanation), so there are only direct spill-over effects.

On the other hand, in the vertical economy there are direct spill-over effects to other firms from a tightening of the constraint of one firm. To understand this more generally, we extend the economy from 3 firms to  $K$  firms and shock the constraint of sector  $i$ . The effects are described in the following proposition.

**Proposition 7.** *(i) In the horizontal economy, there are no direct spill-over effects of tightening constraints. A drop in the constraint of firm  $i$  leads to a fall in firm  $i$  employment and output and an increase in its price. However, there are no direct spill-over effects for the output, employment, and prices of firms  $j \neq i$ .*

(ii) In a vertical economy with  $K$  firms, there are direct spill-over effects following a tightening of a collateral constraint of firm  $i$

for firm  $i$ , employment falls, intermediate input use falls, and output falls. Its equilibrium price rises.

for firms  $i - k$ : employment falls, intermediate input use falls, output falls. Its equilibrium price falls.

for firms  $i + k$ : employment remains unchanged but intermediate input use falls; thus, production falls. Its equilibrium price rises.

We can further summarize these effects in Figure 3.4. Here we solve for the equilibrium in a vertical supply chain economy with 10 firms. We shock the collateral constraint of firm 5 and study what happens in equilibrium to all firms.

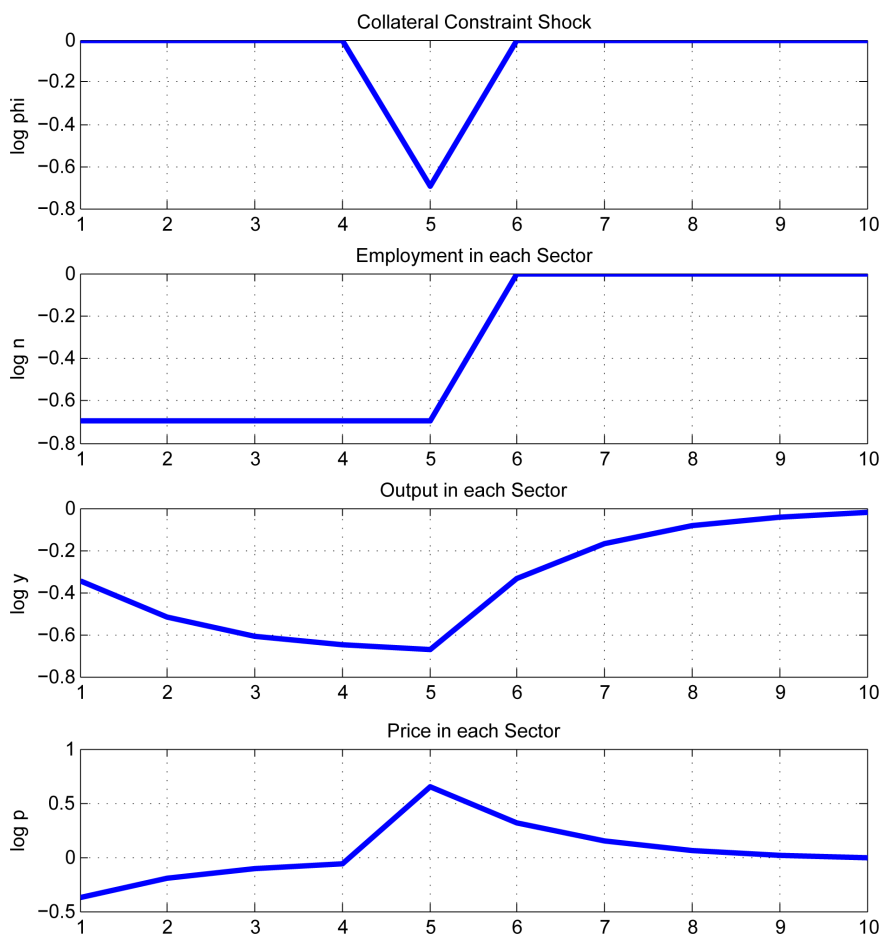


Figure 3: Spill-over Effects

Consider the effects of tightening the collateral constraint of firm 5. This implies that this firm purchases both less labor and less intermediate inputs. Thus, its labor and in-



puts decrease and hence its output falls. Given that its supply decreases, its price thereby increases.

For firm 4, there is now less demand for its good from firm 5. This implies that in equilibrium its output and price falls. In order to produce less output, it both hires less labor and buys less intermediate inputs. Furthermore, this implies that the demand for the good of firm 3 falls. Firm 3 therefore undergoes the same qualitative effects as firm 4: its output, employment, and intermediate input use all falls. But this implies that the demand for the good of firm 2 falls, and so on.

For firm 6, the price of its input (the output of firm 5) is now higher. Thus, it demands less of its intermediate inputs, however, its employment remains unchanged. This firm produces less and because its supply decreases, its price thereby increases.

Thus, there are numerous spill-over effects coming from the tightening of one firm's collateral constraint. In summary, for firms  $i - 1$  there is less demand for their good, so this is like a demand effect. For firms  $i + 1$ , there is an increase in input prices, so this acts like a supply effect. For all firms, these are adverse effects, so that production decreases across the board. One would not see this in a horizontal model nor a representative firm model.

Again, if we had allowed for a different specification of utility such as power utility, we would also have had indirect effects coming from a change in the real wage. However, we've shut these down by using log utility in consumption and linear disutility in labor. If we had allowed for effects coming from the real wage (income and substitution effects) these would be aggregate effects—thereby affecting all firms equally. Therefore, these indirect aggregate effects would have not affected the qualitative results in Proposition 7, and hence without loss of generality we can take them out.

### 3.5 Misallocation

There is a large and growing literature on misallocation in growth and development. Restuccia and Rogerson (2008) show that misallocation of resources across firms can have important effects on aggregate TFP. Banerjee and Duflo (2005) emphasize the importance of resource misallocation in understanding aggregate TFP differences across countries. Furthermore, Hsieh and Klenow (2009) provide quantitative evidence on the potential impact of resource misallocation on aggregate TFP. They present empirical evidence that misallocation across plants may reduce TFP in manufacturing by a factor of two to three in China and India. Jones (2010).. We now consider the question of measuring misallocation in our economy. Here we make the case that standard measures of misallocation are not appropriate for network economies and we propose a new measure which takes into account the location of each firm

within the network.

There are a number of ways to look at misallocation. In general the misallocation literature has looked at the dispersion of marginal revenue products across plants for industries. Hsieh and Klenow (2009) define marginal revenue product of labor as follows

$$MRP_i \equiv \alpha_i \frac{p_i y_i}{n_i} \quad (10)$$

where  $\alpha_i$  is the labor-share of output in firm  $i$ ,  $n_i$  is the amount of labor that firm hires, and  $p_i y_i$  is its revenue.<sup>12</sup>

Hsieh and Klenow (2009) consider something similar to a horizontal economy—firms are heterogeneous and monopolistically competitive. Firm products are aggregated... We now look at the composition of dispersion in marginal products in our economy. Note that in either economy, the following relation holds:  $w_v = \phi_{vi} \alpha_i p_{vi} y_{vi} / n_{vi}$ . That implies that

$$MRP_i = \alpha_i \frac{p_{vi} y_{vi}}{n_{vi}} = \frac{w_v}{\phi_{vi}}$$

If  $\phi_i = 1$  for all firms, then there is no misallocation—marginal revenue products are equalized across all sectors. Now suppose  $\phi_{vi}$  is strictly less than 1 but constant across all firms. We thus reach the following lemma. Thus, MRP is constant across all firms by construction. However, we show that this hides the aggregate distortions that are in fact present.

**Lemma 2.** *Suppose all firms are equally constrained:  $\phi_i = \phi < 1$ . Then, there is no dispersion in  $MRP_i$  in the horizontal economy nor in the vertical economy. However, the horizontal economy is less distorted: it has greater TFP and a greater labor wedge than the vertical economy.*

However, note that in the horizontal economy TFP is greater than in the vertical economy. Thus, at the micro level, if all firms are constrained equally, one would not see any dispersion in marginal products in either economy. However, the vertical economy would have much lower "efficiency" as measured by a greater labor wedge. We also show the same is true if one were to measure this with aggregate TFP. This implies that the misallocation found in Hsieh and Klenow (2009) may in fact be *greater* once one takes into account the network structure of these firms. That is, the horizontal economy provides a *lower bound* for the amount of misallocation generated in a network economy with frictions.

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<sup>12</sup>Hsieh and Klenow (2009) define both Marginal revenue product of labor and Marginal revenue product of capital. Here, since we have only labor, we provide only the definition for marginal revenue product of labor.

Thus, here we argue that  $MRP$ , as defined in (10) is not the right measure to use when thinking about misallocation in network economies with vertical supply chains. We argue for a new measure of misallocation in which we look at the dispersion of the following measure which we call the "Marginal Aggregate Revenue Product"

$$MARP \equiv \tilde{\alpha}_i \frac{PY}{n_i} \quad (11)$$

In our economy, this should be equal to

$$MARP = \tilde{\alpha}_i \frac{PY}{n_i} = \frac{w}{\phi_i}$$

Again, in the frictionless economy, there is no misallocation—the Marginal Aggregate Revenue Product is equated across all firms according to (1):  $\tilde{\alpha}_i \frac{PY}{n_i} = w$ . Thus, there is no dispersion in marginal aggregate products. In the horizontal economy, there again is no misallocation in the sense that there is no dispersion in marginal products. However, in the vertical economy, there is misallocation in the sense that there is dispersion in this measure in marginal aggregate product.

Finally, we can also calculate the marginal products across each of the vertical supply chain firms. An important empirical implication emerges here. In terms of the vertical supply chain, wedges are unambiguously larger for upstream labor, than for downstream labor. Thus, if one has data on the amount of labor used in each sector, and its labor share in aggregate output, one could easily back out the wedges. For example, in the vertical supply chain.

**Proposition 8.** *Suppose all firms are equally constrained:  $\phi_i = \phi < 1$ . Then*

- (i) *in the horizontal economy with frictions, there is no dispersion in  $MARP$ .*
- (ii) *in the vertical economy, there is dispersion in  $MARP$ .*

Thus,  $MARP$  is a better indicator of misallocation in a vertical economy. Furthermore, the marginal aggregate product of labor of firm  $i$  is less than the marginal aggregate product of firm  $i - 1$ .

$$(\phi_{v1}\phi_{v2}\phi_{v3})\tilde{\alpha}_1\frac{Y_v}{n_{v1}} = (\phi_{v3})\tilde{\alpha}_3\frac{Y_v}{n_{v3}}$$

This is one empirical implication that one may be able to look for in the data. Not sure whether we should put this in a proposition or not.

### 3.6 Summary

In this section we studied two economies, a horizontal economy in which firms did not transact with each other and a vertical economy in which firms were arranged in a supply chain. These economies were allocationally equivalent under no frictions. We found that financial frictions then drove wedges between each firm's marginal benefit and marginal cost of production. However, due to the different economy structures this lead to different wedges between marginal aggregate product... We summarize our main results from this simple model as follows.

*Result 1.* Pledgeability constraints introduce wedges between the marginal product of labor and marginal cost, however these wedges differ across the two economies. Aggregate labor wedges are simply a weighted sum of these individual wedges.

*Result 2.* For any allocation, more liquidity is needed in the vertical economy than in the horizontal economy to implement this allocation.

*Result 3.* In vertical economies, the most downstream firm has the greatest impact on aggregate output.

*Result 4.* The liquidity multiplier in the vertical economy is greater than the liquidity multiplier in the horizontal economy. Tightening constraints in the vertical economy leads to a greater drop in output than in the horizontal economy.

*Result 5.* There are no direct spill-over effects in the horizontal economy. However, there are direct spill-over effects in the vertical economy. The collateral constraint acts like an adverse demand shock on firms upstream ( $i - k$ ), while it works as an adverse supply shock on firms downstream ( $i + k$ ).

*Result 6.* When firms are arranged in production networks with vertical chains, the right measure of misallocation should be the dispersion in marginal aggregate products across firms, not the dispersion in marginal individual products. If firms were equally constrained, the former measure would pick up misallocation in the vertical economy, whereas the latter measure would not.

All of these results suggest that looking at network structures is important when thinking about the impact of financial frictions.

## 4 The General Production Network

We now consider a more general environment. We follow Acemoglu, Carvalho Ozdaglarz, and Tahbaz–Salehi (2012) and Jones (2011) in working with a general input-output structure, which is essentially a static version of Long and Plosser (1983).

### 4.1 The Model

The economy is populated by a representative household and  $n$  production sectors. Each production sector consists of a continuum of competitive firms.

*Households.* The representative household is endowed with one unit of labor, supplied inelastically. The household consumes the  $n$  differentiated goods produced by the production sectors. Its preferences over these goods are Cobb-Douglas and given by the following<sup>13</sup>

$$U(c_1, c_2, \dots, c_n) \equiv \prod_{i=1}^n (c_i)^{1/n}.$$

where  $c_i$  is the consumption of good  $i$ . The budget constraint of the household is given by

$$\sum_{i=1}^n p_i c_i = h \tag{12}$$

where  $p_i$  is the price of good  $i$  and  $h$  is the wage.

*Production.* Each good in the economy is produced by firms within a competitive sector. Goods are differentiated across sectors, but not across firms within a sector. The production of any given sector can either be consumed or used by other sectors as an input (intermediate goods) for production. From here on we drop the terminology of firms and just think of sectors as the production unit, with the understanding that it is perfectly competitive.

Sectors produce output using labor and intermediate goods inputs. They have Cobb-Douglas technologies with constant returns to scale given by the following

$$x_i = z_i \ell_i^\alpha \left( \prod_{j=1}^n x_{ij}^{\omega_{ij}} \right)^{1-\alpha} . \tag{13}$$

where  $x_i$  is the output of sector  $i$ ,  $\ell_i$  is the amount of labor hired by that sector,  $\alpha \in (0, 1)$  is the share of labor,  $x_{ij}$  is the amount of commodity  $j$  used in the production of good  $i$ , and

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<sup>13</sup>In many ways the household is treated as a sector, hence the use of  $x$  to denote consumption and  $y$  to denote labor (output).

$z_i$  is the idiosyncratic sector-specific productivity shock.<sup>14</sup> We assume that productivity  $z_i$  is drawn from known distribution  $F_i$ . Finally, the exponent  $\omega_{ij}$  designates the share of good  $j$  in the total intermediate input use of firms in sector  $i$ . We assume that  $\sum_{j \in N_i} \omega_{ij} = 1, \forall i \in \{1, \dots, N\}$ . This implies that the sectoral production functions all exhibit constant returns to scale.

Sector  $i$  maximizes profits given by

$$\Pi_i = \max_{\sigma_i, x_i} p_i x_i - h l_i - \sum_{j=1}^n p_j x_{ij}$$

subject to its financial constraint given by<sup>15</sup>

$$h l_i + \sum_{j=1}^n p_j x_{ij} \leq \phi_i p_i x_i$$

That is, similar to as we had before, the expenditure of sector  $i$  is constrained to be less than  $\phi_i$  of its earnings.

*Market Clearing.* The resource constraints are given by

$$x_i = c_i + \sum_{j=1}^n x_{ji}, \forall i \in \{1, \dots, N\} \quad (14)$$

and for labor

$$\sum_{i=1}^n l_i = 1$$

*Input-Output Matrix and Notation.* We let  $W$  denote the input-output matrix of the economy with entries  $w_{ij}$ :

$$W = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & & \\ \vdots & & & \\ w_{n1} & & & w_{nn} \end{bmatrix}$$

We also adopt the convention that  $w_{ij} = 0$  if sector  $j$  is not an input supplier to sector  $i$ . Note that the rows of  $W$  sum up to one due to the constant returns-to-scale assumption. However the columns need not sum to 1, and in fact their sum is what is known as the weighted outdegree. This corresponds to the share of sector  $i$ 's output in the input supply

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<sup>14</sup>In general,  $\omega_{ii}$  need not be equal to 0; sectors may use their own product as an input. This is in fact shown to be true in the data.

<sup>15</sup>As before, the limited enforcement problem is described in Appendix A.

of the entire economy. Input-output relationships between different sectors can equivalently be represented by a directed weighted graph on  $n$  vertices, each corresponding to a particular sector of the economy.

## 4.2 Equilibrium Definition and Characterization

We now define an equilibrium for this economy as follows.

**Definition 4.** *A competitive equilibrium consists of a vector of prices  $(p_1, p_2, \dots, p_n)$ , a wage  $h$ , a consumption bundle  $(c_1, c_2, \dots, c_n)$ , input, output and labor allocations  $(x_i, \ell_i, \{x_{ij}\}_{j=1}^n)$  such that (a) the representative household maximizes utility (b) the representative firms in each sector maximize profits, and (c) prices clear commodity markets and the wage clears the labor market.*

We now move on to characterizing the equilibrium.

*Household's Problem.* The household's optimization problem is fairly straight-forward. The FOCs of the household's problem imply that household expenditure on individual goods is equal across all goods

$$p_i c_i = p_j c_j, \quad \forall i, j$$

Substituting this back into the household budget constraint (12), we have that expenditure on any good is equal to  $1/n$ th of household income.

$$p_i c_i = h/n, \quad \forall i$$

*Firm's Problem.* The representative firm's profit maximization problem is given by

$$\max p_i z_i \ell_i^\alpha \left( \prod_{j=1}^n x_{ij}^{\omega_{ij}} \right)^{1-\alpha} - h \ell_i - \sum_{j=1}^n p_j x_{ij}$$

subject to the firm's pledgeability constraint

$$h \ell_i + \sum_{j=1}^n p_j x_{ij} \leq \phi_i p_i x_i$$

It's conceptually easier to first do a cost minimization problem. Consider a firm which chooses to minimize cost subject to producing  $x_i$  of the composite. The optimal input use problem is given by,

$$c(x_i) = \min_{x_{ij}} h \ell_i + \sum_{j=1}^n p_j x_{ij} \tag{15}$$

subject to

$$x_i = z_i \ell_i^\alpha \left( \prod_{j=1}^n x_{ij}^{\omega_{ij}} \right)^{1-\alpha}$$

Let  $\lambda_i$  be the Lagrange Multiplier associated with the constraint. The first order conditions from this problem imply

$$\frac{h\ell_i}{\alpha} = \frac{p_j x_{ij}}{(1-\alpha)\omega_{ij}} \quad \text{and} \quad \frac{p_j x_{ij}}{p_k x_{ik}} = \frac{\omega_{ij}}{\omega_{ik}}, \quad \forall j, k \quad (16)$$

From this, we have that the share of firm  $i$ 's expenditure on the input produced by firm  $j$  is proportional to its share in production. One can substitute this relationship into the the cost function (15) to obtain the relation between the sector  $i$ 's intermediate input and the  $j$  sector input,

$$c(x_i) = \frac{1}{\alpha} h \ell_i \quad \text{and} \quad c(x_i) = \frac{1}{(1-\alpha)\omega_{ij}} p_j x_{ij} \quad (17)$$

Finally, the firm's financial constraint implies that  $c(x_i) = \phi_i p_i x_i$ . This implies

$$\begin{aligned} \ell_i &= \phi_i \alpha \frac{1}{h} p_i x_i \\ x_{ij} &= \phi_i (1-\alpha) \omega_{ij} \frac{1}{p_j} p_i x_i \end{aligned}$$

Substituting these values back into the production function (13) and taking logs gives us

$$\begin{aligned} \alpha \log h &= \log z_i + \log \phi_i + \alpha \log \alpha + (1-\alpha) \log (1-\alpha) + \log p_i \\ &+ (1-\alpha) \sum_{j=1}^n \omega_{ij} \log \omega_{ij} - (1-\alpha) \sum_{j=1}^n \omega_{ij} \log p_j \end{aligned} \quad (18)$$

Let us define the following vectors

$$\mathbf{z} = \begin{bmatrix} \log z_1 \\ \log z_2 \\ \vdots \\ \log z_n \end{bmatrix} \quad \text{and} \quad \boldsymbol{\phi} = \begin{bmatrix} \log \phi_1 \\ \log \phi_2 \\ \vdots \\ \log \phi_n \end{bmatrix}, \quad \text{and} \quad \mathbf{p} = \begin{bmatrix} \log p_1 \\ \log p_2 \\ \vdots \\ \log p_n \end{bmatrix}$$

Then stacking equation (18) upon one another where each row  $i$  corresponds to equation



(18) for sector  $i$ , we have that

$$\mathbf{1}\alpha \log h = (\mathbf{z} + \boldsymbol{\phi}) + (I - (1 - \alpha)W)\mathbf{p} + \mathbf{B}$$

where  $\mathbf{1}$  is a column vector of  $n$  ones and

$$\mathbf{B} = \mathbf{1} [\alpha \log \alpha + (1 - \alpha) \log (1 - \alpha)] + (1 - \alpha) \begin{bmatrix} \sum_{j=1}^n \omega_{1j} \log \omega_{1j} \\ \sum_{j=1}^n \omega_{2j} \log \omega_{2j} \\ \vdots \\ \sum_{j=1}^n \omega_{nj} \log \omega_{nj} \end{bmatrix}$$

Multiplying this equation by the  $i$ -th element of the vector  $(\frac{1}{n}(I - (1 - \alpha)W')^{-1}\mathbf{1})'$  gives us

$$\log h = v'(\mathbf{z} + \boldsymbol{\phi}) + \frac{1}{n}\mathbf{1}'\mathbf{p} + \mathbf{v}'\mathbf{B}$$

or

$$\log h = v'(\mathbf{z} + \boldsymbol{\phi}) + \mu$$

where  $\mu$  is a constant independent of the vector of shocks and is given by

$$\mu = \frac{1}{n}\mathbf{1}'\mathbf{p} + \mathbf{v}'\mathbf{B}$$

Note that

$$\frac{1}{n}\mathbf{1}'\mathbf{p} = \frac{1}{n} \sum_{i=1}^n \log p_i$$

corresponds to the ideal price index  $n(p_1 p_2 \cdots p_n)^{1/n}$  which we may normalize to be equal to 1.

*Equilibrium Characterization.* Finally, we calculate aggregate gdp in this economy. Real value added in the economy is  $\sum_{i=1}^n p_i c_i = h$ . This implies that GDP is given by  $y = \log(GDP) = \log h$ .

**Proposition 9.** *In the competitive equilibrium of this economy, the logarithm of GDP is given by*

$$y = \log(GDP) = v'(\mathbf{z} + \boldsymbol{\phi}) + \mu$$

where  $\mu$  is a constant and  $v$  is what is known as the influence vector, defined as

$$v \equiv \frac{1}{n}(I - (1 - \alpha)W')^{-1}\mathbf{1} \tag{19}$$

Thus, equilibrium aggregate output is simply a log-linear function of the underlying productivity and collateral shocks in the economy. The weights or coefficients on these shocks are summarized by the influence vector, which captures how sectoral productivity shocks propagate to other sectors of the economy and ultimately affect aggregate output. This highlights the fact that sectors that have positions within the network with more direct or indirect downstream customers play a larger role in driving aggregate volatility.

We can compare our results to that of Acemoglu et al (2011). In their paper without financial frictions, they obtain the following representation of aggregate GDP.

$$y = \log(GDP) = v'z + \mu$$

Thus, aggregate GDP is simply a log-linear function of underlying productivity shocks, where the influence vector  $v$  is defined similarly as in (19). This implies that conditional on knowledge of the influence vector  $v$ , a productivity shock is isomorphic to a collateral shock in our model, in terms of its effect on aggregate output.

However, a simplifying result in Acemoglu et al. (2011) is that the influence vector  $v$  may be represented as the “sales vector” of the economy. In particular, Acemoglu et. al (2011) show that the influence vector simply captures the equilibrium share of sales of different sectors as follows

$$v_i = \frac{p_i x_i}{\sum_{j=1}^n p_j x_j}$$

This implies that if one were to have data simply on the sales in each sector, one could easily calculate the influence vector without necessarily knowing the underlying network of the model.

We show that this simplification, however, is not true in our model. When there are financial frictions, we find that equilibrium sales in fact is not equal to the influence vector.

**Lemma 3.** *With financial frictions, the influence vector is not equal to equilibrium sales*

$$v_i \neq \frac{p_i x_i}{\sum_{j=1}^n p_j x_j}$$

*In particular, equilibrium sales are given by*

$$s' = \frac{h}{n} \mathbf{1}' (I - (1 - \alpha) \Phi' W)^{-1}$$

where

$$\Phi = \begin{bmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2 & & \\ \vdots & 0 & \ddots & \\ 0 & & & \phi_n \end{bmatrix}$$

This implies that equilibrium sales depend not only on the input-output structure, but also on the financial frictions—the  $\phi$ 's. Hence, equilibrium sales are not a sufficient statistic for computing the influence vector  $v$ . For this, it is then necessary that one have information on the underlying input-output matrix  $W$ .

### 4.3 Generalization of previous results

First, there is an isomorphism here to the tax literature. Note also that the solution to the firm's problem is also characterized by the same first order condition to the problem of firm that faces a sales tax, or equivalently to a firms facing input taxes. It is convenient then to characterize the firm's problem in terms of a firm facing sales taxes. To be more precise, a firm faces a sales taxes maximizes profits:

$$\max_{x_i} (1 - \tau_i) p_i x_i - h \ell_i - \sum_{j=1}^n p_j x_{ij}$$

The solution to this problem produces a wedge.. Thus, the firm's problem is equivalent to a the problem of a firm facing a sales tax of  $(1 - \tau_i) \alpha_i = \phi_i$ . Thus, the corresponding tax for firm  $i$  is  $\tau_i \equiv 1 - \frac{\phi_i}{\alpha_i}$ . This tax has an immediate interpretation. At the unconstrained optimal,  $\alpha_i$  is the fraction of the firm's revenue that is needed to pay for inputs, or 1 minus the markup.  $\chi_i$  represents the fraction of revenues that can be obtained as the firm's credit. This credit is the sum of the firm's liquid funds  $\omega_i$  and the fraction that can be obtain directly as trade credit  $\theta_i$ . Thus, the tax is the fraction between the firm's liquidity needs and its actual li

Furthermore, we establish the following isomorphism to the tax literature.

**Lemma 4.** *An equilibrium allocation is equivalent to the equilibrium allocation of an economy with sales taxes, and lump-sum transfers where the sales tax in sector  $i$ , is given by:*

$$1 - \tau_i = \frac{\phi_i}{\alpha_i}.$$

Thus, the environment is isomorphic to the input-output model with distortions of Jones (2011).

## 4.4 Financial Shocks in Sample Economies

We compare the response of GDP to a tailored sequence of liquidity shocks that are common to every sector.<sup>16</sup> The exercise allows us to compute a liquidity multiplier for the 5 sample economies.

Figure 4.4 shows the behavior of GDP in the model for various network structures to the same liquidity shock sequence. The figure is divided into 5 rows, each corresponding to a different production structure. The boxes on the left of each row are the graphical representations of the I-O matrix in each model. The top row for example, corresponds to the case of a horizontal network. No sector uses any other input but labor which is why the box has a solid color. The liquidity multiplier is approximately one and the sequence of GDP (solid line) tracks almost perfectly the sequence of the liquidity shock. The intuition has been discussed earlier in the paper. With a horizontal structure, firms operate in isolation and the liquidity shock directly constrains the amount of workers each firm hires. The percentage drop in liquidity, to a first order, is equivalent to the percent reduction in hours and roughly the labor share of GDP. However, there is an indirect effect that operates through the reduction in wages that reduces total hours worked. The multiplier of 1 turns out as a result of the labor share and the Frisch elasticity. The following cases are a triangular production chain, a vertical chain, a production cycle, and a star network. These sample economies are organized in increasing order of their impact. Note that in all cases, the multiplier is about twice as large. It shouldn't be a surprise that the star network yields the largest impact. However, it is a surprise that its multiplier is roughly close to the multiplier in all of the other models with a sectoral linkage. The reason is that we are studying an aggregate shock. It is worth discussing what is going on at the cross-section level of each model. We do this for selected network structures of size  $N=10$ . In all examples, households demand goods from every sector.

*Horizontal.* Figure 4.4 shows the cross sectional impact of liquidity shocks in a horizontal economy. The top left panel once again the I-O matrix. The three panels to the right describe the cross-section effects on the GDP of different sectors to an increase in the liquidity of a single sector. For example, the second panel shows (red stem) a reduction of liquidity in sector 1 of 1%. This shock induces a reduction in that sector's GDP of 1.8%. Since the horizontal economy has no spill over effects beyond an indirect effect via wages in the labor market, the impact on the GDP of other sectors is mute. The next panel shows a symmetric response to a shock in sector 6 and the right panel to sector 10. The bottom panel presents

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<sup>16</sup>The sequence of liquidity shocks is chosen so that when we calibrate the model to match the I-O matrix of the U.S. economy, we obtain a pattern that replicates aggregate output during a chosen sample period.

the response of GDP as we shock each sector by 1%. Since this economy is symmetric, what sector is hit doesn't matter for the aggregate response.

*Circle.* Figure 4.4 repeats the information of Figure 4.4 for a circle production network economy. As can be noted from the top left panel, in this economy, sector  $i$  requires inputs from sector  $i - 1$  for all  $i > 1$  and  $i = 1$  requires inputs from sector 10. A first thing to note is that this network has similar cross-sectional properties as the horizontal economy because its symmetric (bottom panel). However, the top panels show that here spill-over effects are present. Sectors that supply inputs to the affected sector are affected with a negative reduction in their GDP. The sectors affected will experience an increase in their price. Sectors that require inputs from those sectors will substitute that input with labor, something that can be done here because labor is substitutable with the production input. The supplied sector's output can actually expand because they benefit from lower wages and take over part.

*Vertical.* A vertical economy's response is quite different in Figure 4.4. In the vertical economy is no longer symmetric. In fact, it matters a lot what sector is being affected for aggregate GDP and the cross-sectional impact. Here sectors of lower  $i$  supply inputs to sectors of higher  $i$  with sector 1 not hiring inputs other than labor. The bottom panel shows that hitting the most downstream sector is more important. This was discussed earlier in the paper. Downstream firms have the largest impact. Also, spill-over effects matter a lot for the sector that is being affected. It is interesting to observe that when the last sector is hit, the firms upstream experience an expansion in their output because they supply goods to consumer who substitute consumption from the last sector to lower  $i$  sectors. When the first sector is affected by liquidity, it will affect the supply of inputs to the sector above it and that sector to the sector above it. This causes a chain reaction with reductions in GDP in these sectors. Output of downstream firms can actually expand because of the reduction in labor.

*Triangular.* If upstream firms are supplying more inputs to an even larger number of firms downstream, the observation that liquidity matters more downstream reverts. This is observed in Figure 4.4. It is worse to affect liquidity of the key supplier. Note also that the chain effect affects an even larger number of sectors than in the vertical economy.

*Star.* Finally, we examine a star network where every sector requires the input of a key sector, a sector like utilities or transportation in the real world. In Figure 4.4, that key sector is sector 1. This sector's shock has a much larger impact on aggregate output. All other sectors have a minor impact because they don't affect the supply of intermediate inputs for other sectors. In fact, shocking the star in the network has the effect of an aggregate shock in the vertical economy, something that is known in the networks literature.

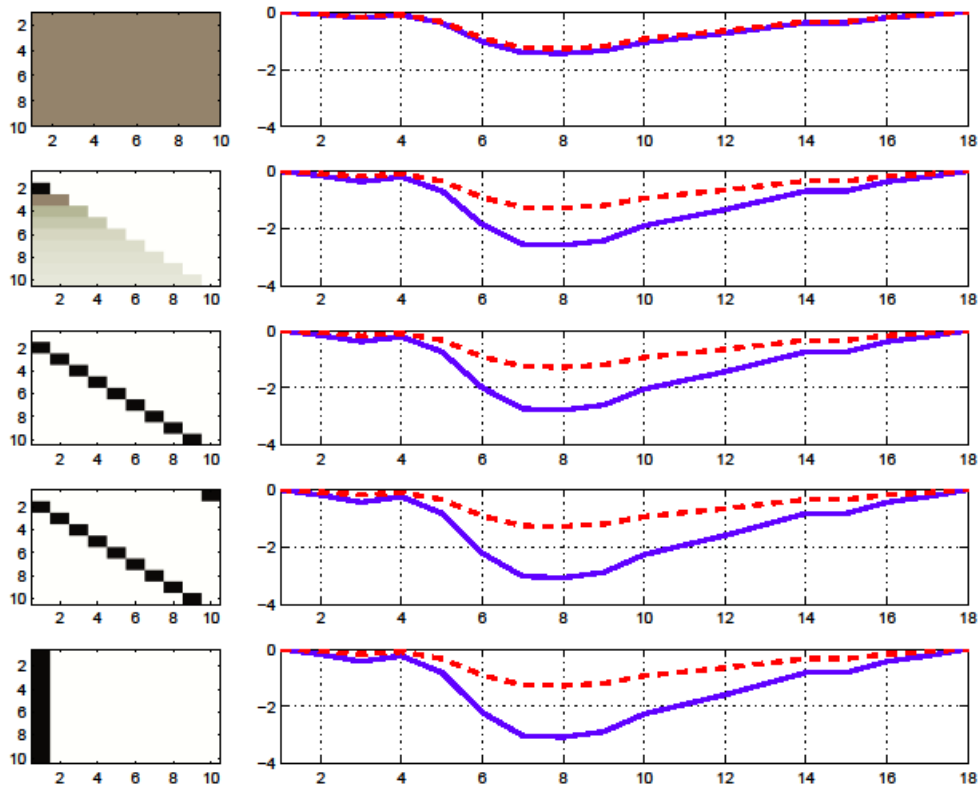


Figure 4: Impact of Liquidity in different Production Networks

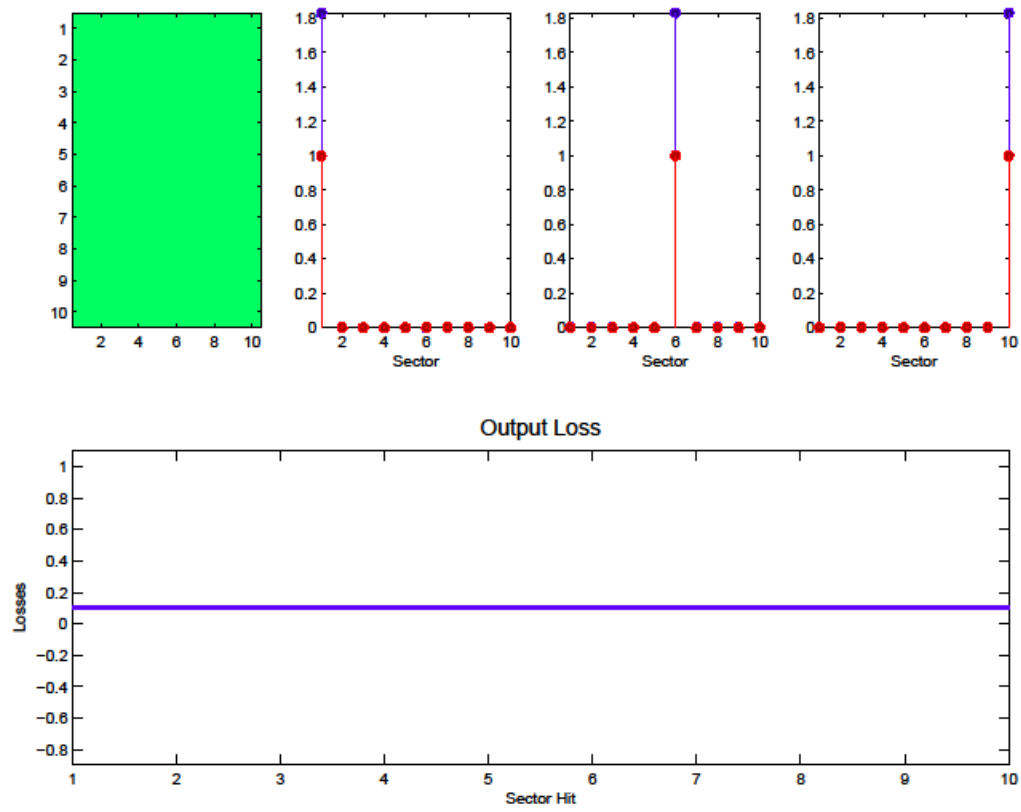


Figure 5: Impact of liquidity in horizontal network

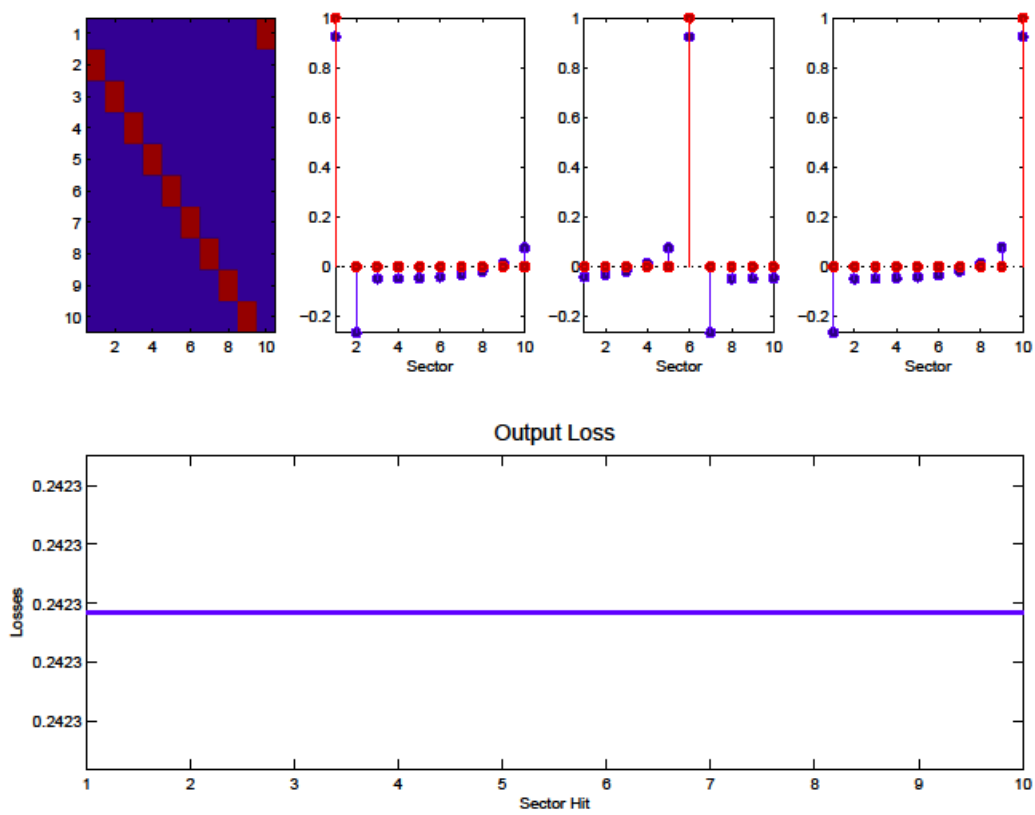


Figure 6: Impact of liquidity in circle network



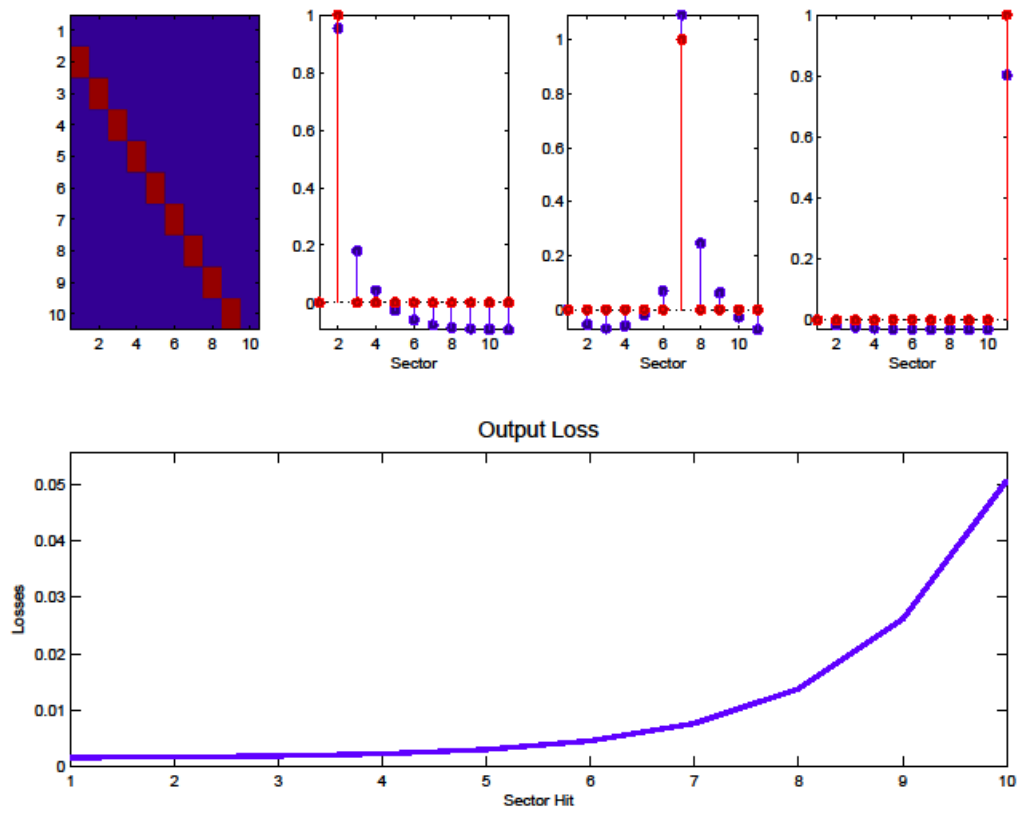


Figure 7: Impact of liquidity in vertical network

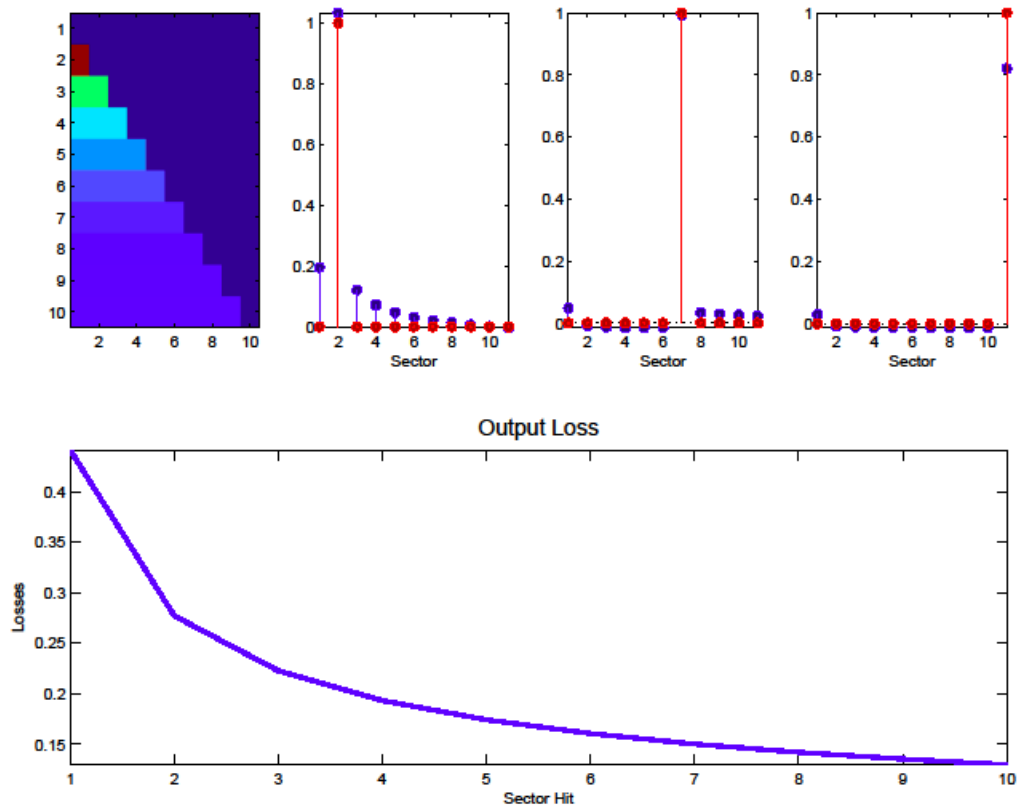


Figure 8: Impact of liquidity in triangular network

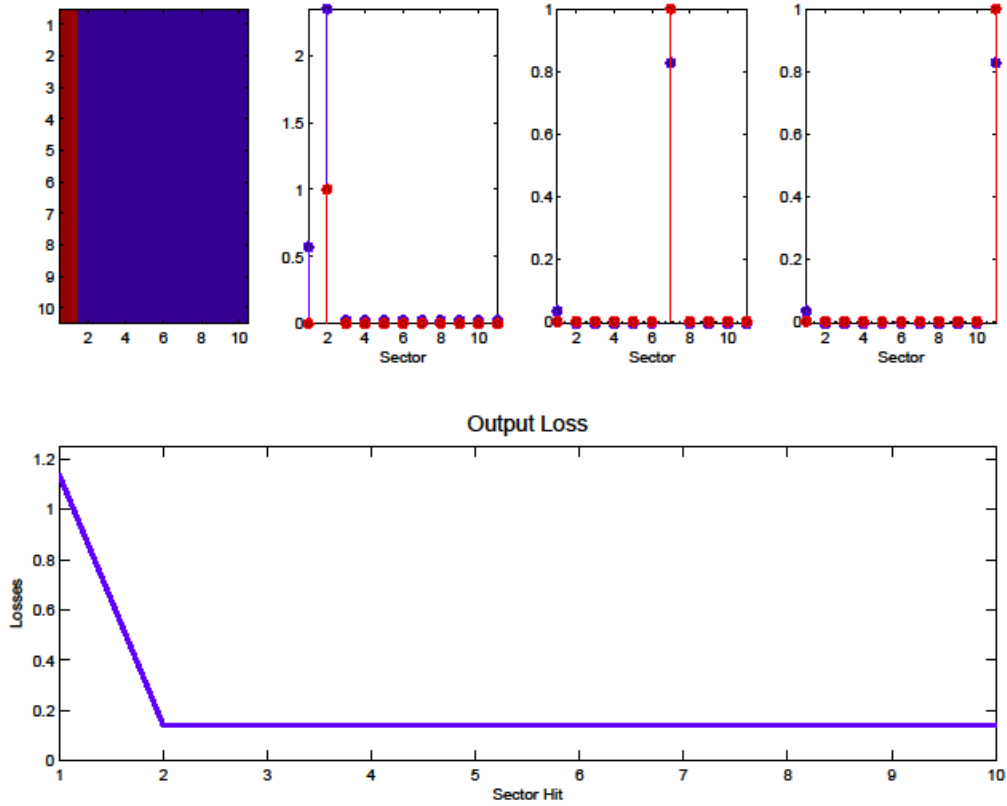


Figure 9: Impact of liquidity in star network

## 5 Calibration

In this section, we calibrate the general production network described in Section 4 to the input-output matrix of 3-digit level sectors of the U.S. economy.

Idea for calibration: can back out wedges. Similar to Chari-Kehoe-McGrattan business cycle accounting. But instead, sectoral accounting with networks.

That is, the elasticity of the response of aggregate GDP to a reduction in liquidity of any or all sector's of the economy. The theoretical analysis presented earlier shows that when production is organized horizontally and labor is inelastically supplied, that multiplier is 0. If instead, labor is supplied elastically, the response depends on the labor supply elasticity and the elasticity of output to labor. However, production in the United States is organized via a highly interconnected network. This section shows that calibrating the input output matrix of our model to that of the U.S. economy, the multiplier turns out to be 3.8. We

will explain how we estimate this number and what additional empirical implications our model has. We organize this section into ten questions. The first set of questions are about aggregates and the rest correspond to sectoral analysis. We begin discussing the calibration.

For our empirical analysis, we set household's parameters to  $\gamma = 2$  and  $\varepsilon = 1/2$  which is standard in macro calibrations. We use the Input-Output accounting tables reported by the Beareau of Economic Analysis (BEA) to calibrate the rest of the model.

The BEA organizes the summary level of Input-Output matrixes at a three-digit level according to the North American Industry Classification System (NAICS). There are 65 NAICS sectors at three-digit level. We collect yearly data from 1998 until 2011. The Use of Commodities by Industries after Redefinitions (USES) table is used to calibrate the  $\alpha_{oj}$  shares of consumption. This table reports the expenditures in US\$ billions of all industries and households for every commodity produced in every industry. Thus, we the following property of our model to calibrate  $\alpha_{oj}$  :

$$\alpha_{oj} = \frac{p_j x_{0j}}{\sum_{n=1:65} p_n x_{0n}}.$$

Let  $u_{ij,t}$  be the US\$ Billion expenditures by sector i in sector j's commodities. This data is the corresponding analog for the product  $p_j x_{ij}$  in our model. The USES table reports personal consumption expenditures (Column F010 of that table). Thus,  $p_j x_{0j}$  is reported so we compute the consumption share of sector j's commodity at time  $t$ ,  $\alpha_{oj,t}$  by

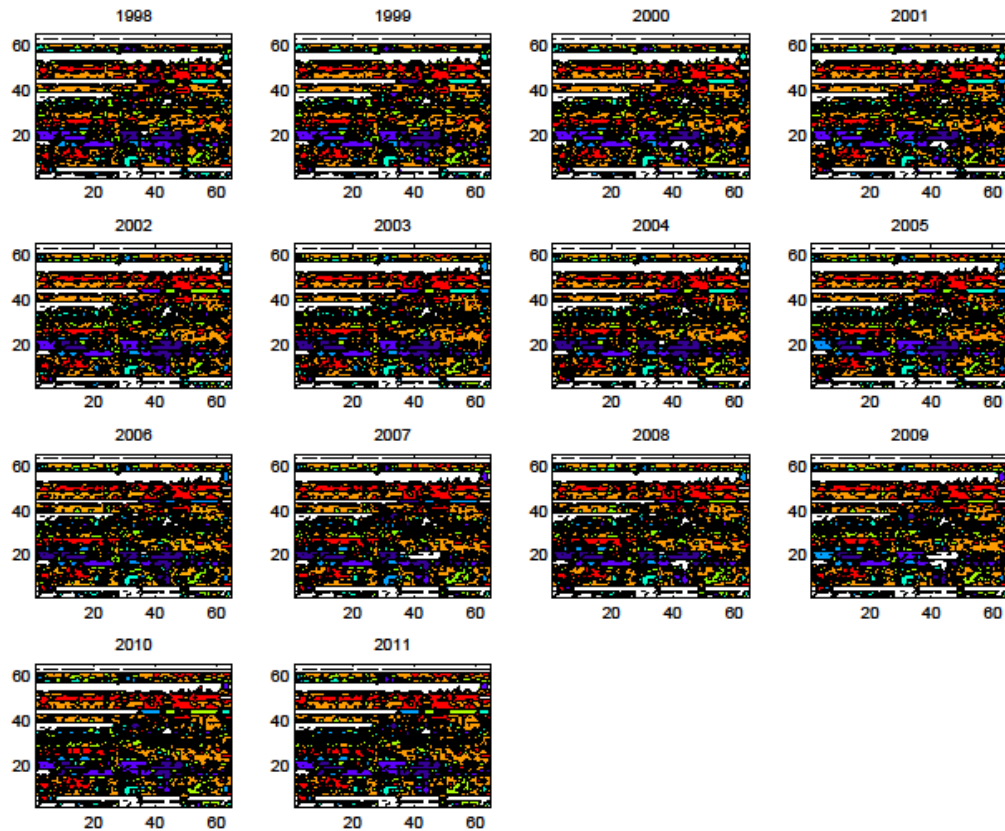
$$\alpha_{oj,t} = \frac{u_{oj,t}}{\sum_{n=1:65} u_{on,t}}$$

Figure ?? reports the evolution of these shares for the 1998-2011 showing a very stable behavior for most sectors. The exceptions are mining and petroleum industries which happen to have the lowest values to begin with.

The technology parameters are directly calibrated from the Input-Output Matrix. Each columns of the Matrix reports the fraction of expenditures of each sector create a dollar of sales in that sector. Let,  $\alpha_{ij,t}^{Data}$  represent the share of sector's i expenditures of on j sector commodities over sector i's sales. Since the matrix columns includes added value and taxes, we calibrate the coefficients by normalizing by expenditures on inputs and labor. Thus we leave outside 3 entries: (1) scrap, used and secondhand goods, (2) non-comparable imports and rest-of-the-world adjustment and (3) taxes on production and imports, less subsidies without including taxes. The coefficients of the production function are obtained by setting:

$$\alpha_{ij,t} = \frac{\alpha_{ij,t}^{Data}}{\sum_{n=1:65} \alpha_{ij,t}^{Data} + \alpha_{0j,t}^{Data}}.$$

The input-output matrix presents a highly stable pattern. Figure 5 shows the contour plot for the log of the entries of the matrix during the years of the sample 1998:2011.



We cannot compute the coefficients of decreasing returns to scale of sector  $i$  at period  $t$ ,  $\alpha_{i,t}$  directly because our model yields predictions only about the fraction of costs. Thus, we obtain an estimate of  $\chi_{i,t}$

$$\chi_{i,t} = 1 - \left( \sum_{n=1:65} \alpha_{ij,t}^{Data} + \alpha_{0j,t}^{Data} \right).$$

These shares are very unstable compared to the expenditure shares of consumption in the U.S.. This pattern can be seen from Figure ???. Through the lens of the model, the instability of costs/sales of each sector is purely the outcome of liquidity shocks. We will treat  $\chi_{i,t}$  as our primitives. We cannot identify the term  $\alpha_{i,t}$  from our observation of  $\chi_{i,t}$ . This is the

main drawback of the paper. Because we are aware of this inconvenience, for the rest of the section we adopt a cautious approach. That is, we will make assumptions that minimize the impact of liquidity by assuming that  $\alpha_{i,t}$  is as low as possible. We are now ready to answer some questions.

## 5.1 Aggregate Analysis

**Question 1. Suppose liquidity is drawn down by in 1% in every sector. What is the multiplier?** By 3.80%. The answer to this question depends on the parameters that we input to the model that in turn, depend on a given year's input output matrix. To compute the number, we numerically approximate the derivative of output to a reduction in liquidity in every sector. The corresponding directional derivative is  $\langle \frac{\partial y(\Theta_t)}{\partial \chi_{i,t}}, I \rangle$  computed along the unit vector. Since the derivative depends on the calibration for a given year, we average across all years of our sample.

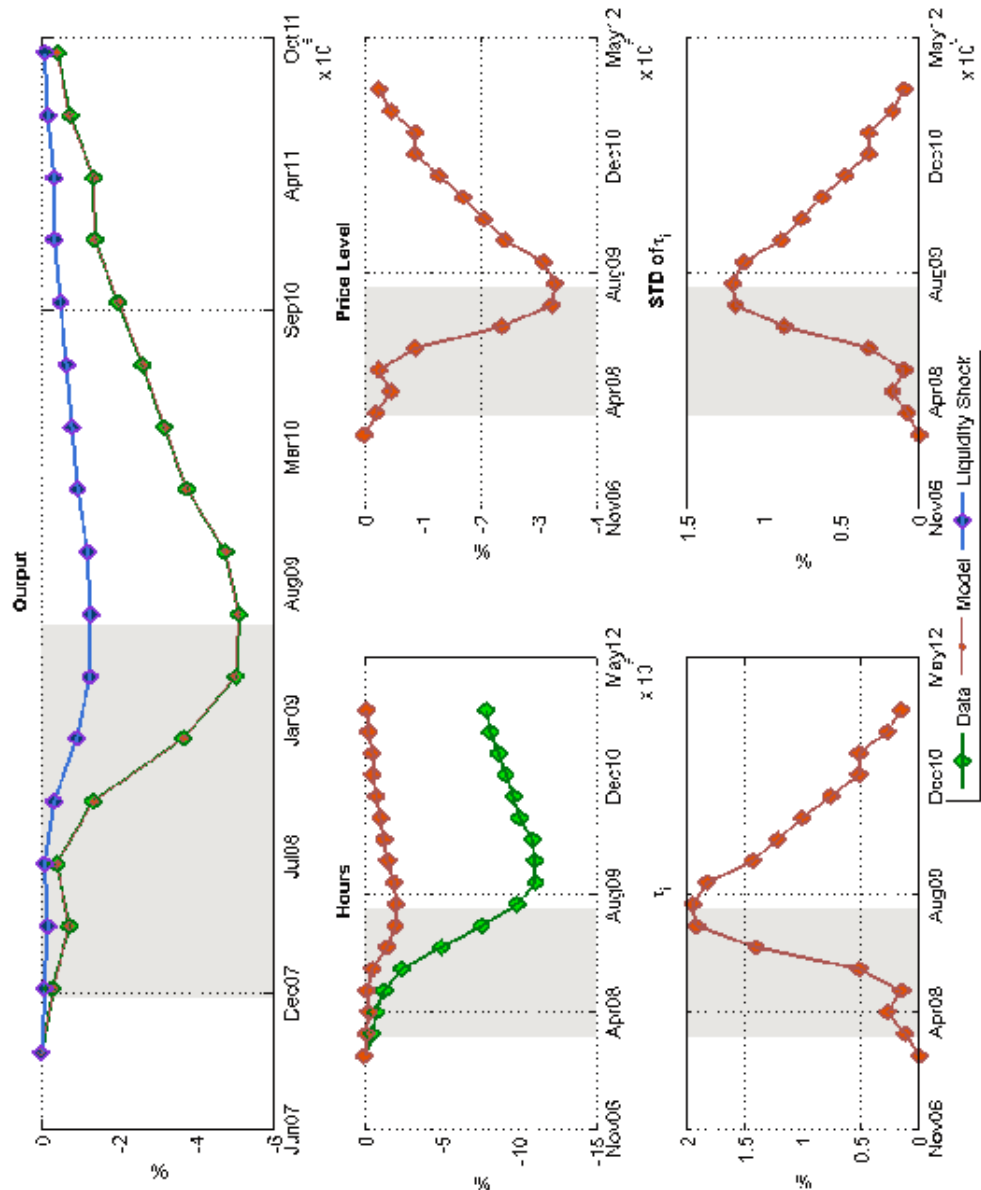
$$\mu = \sum_{t=1998:2011} \frac{\langle \frac{\partial y(\Theta_t)}{\partial \chi_{i,t}}, I \rangle}{y_{i,t}(\Theta_t)} = 3.8$$

The range of answers for given years goes from 3.5 to 4.

**Question 2. What is the aggregate model implied liquidity shock that could have caused the recession?** Figure 5.1 reports the response of GDP together with several equilibrium objects to a particular sequence of liquidity shocks. The sequence corresponds to the quarters ranging from the third quarter of 2007 until the second quarter of 2011. The sequence is chosen so that GDP in the model follows the same path as the deviations from its the average growth rate of the U.S. economy over that last 30 years. The top panel reports the reduction in output, both for the model and the data which by construction coincide. The same panel also reports the aggregate liquidity shock that hits every sector symmetrically that would generates the path for GDP. At the trough of the cycle, the required reduction in liquidity would have been of 1.28% of sales for every sector. A model with a representative firm would require at least 7.6% reduction in liquidity.<sup>17</sup> Now, it is important to note that in our model, this is coming mostly from a production missallocation effect. The data shows a much higher reduction in labor and much lower adjustment in prices. Possibly, nominal rigidities would magnify the multiplier.

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<sup>17</sup>Consider the case where wages are fixed. if Labor is the only factor for production and GDP falls by  $\Delta y\% = 5\%$  away from its linear trend at the trough of the cycle, the model requires  $\frac{1}{\alpha}\Delta y\%$  which calibrated at  $\alpha = 2/3$  yields the quoted number



**Question 3. How large are spill-over effects?** A natural question at this stage is to ask, how much of the reduction in GDP has to do with the spill-over effects described in the previous answer. Spill-over effects will show up in the price schedule faced by the firm. In the analysis of a vertical economy we argued that the sales price of a firm will face a force

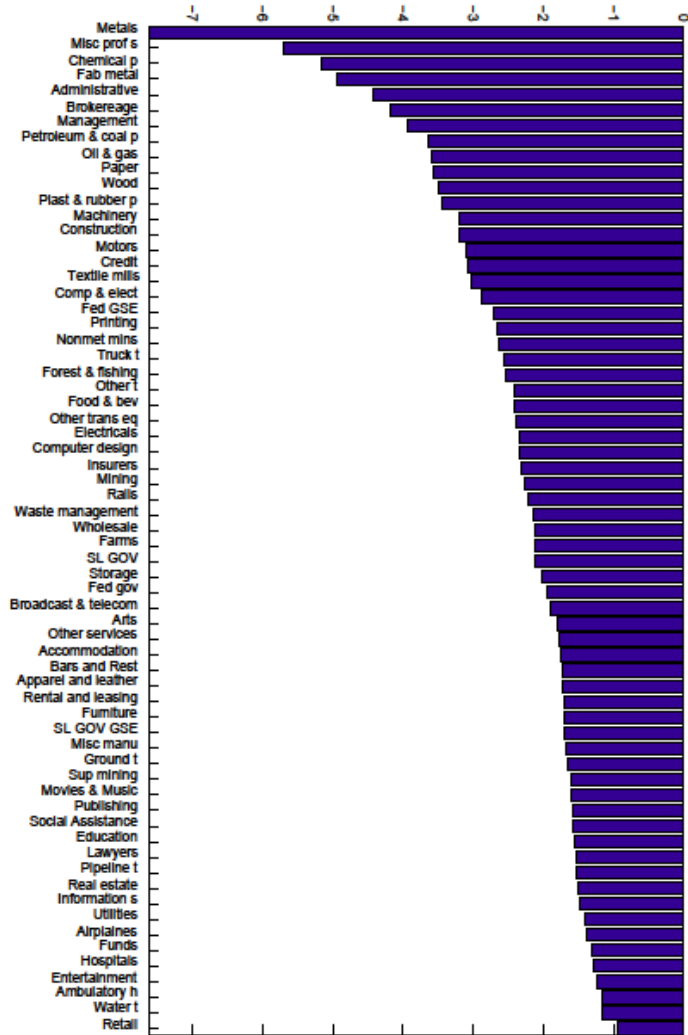
that pushes it downwards when other sectors downstream are constrained. Similarly, the purchase cost of commodities produced by sectors upstream that face a constraint should increase in this environment. Indeed, such mechanisms are operating in the hypothetical exercises we are studying here. One statistic that captures this missallocation is obtained by the following:

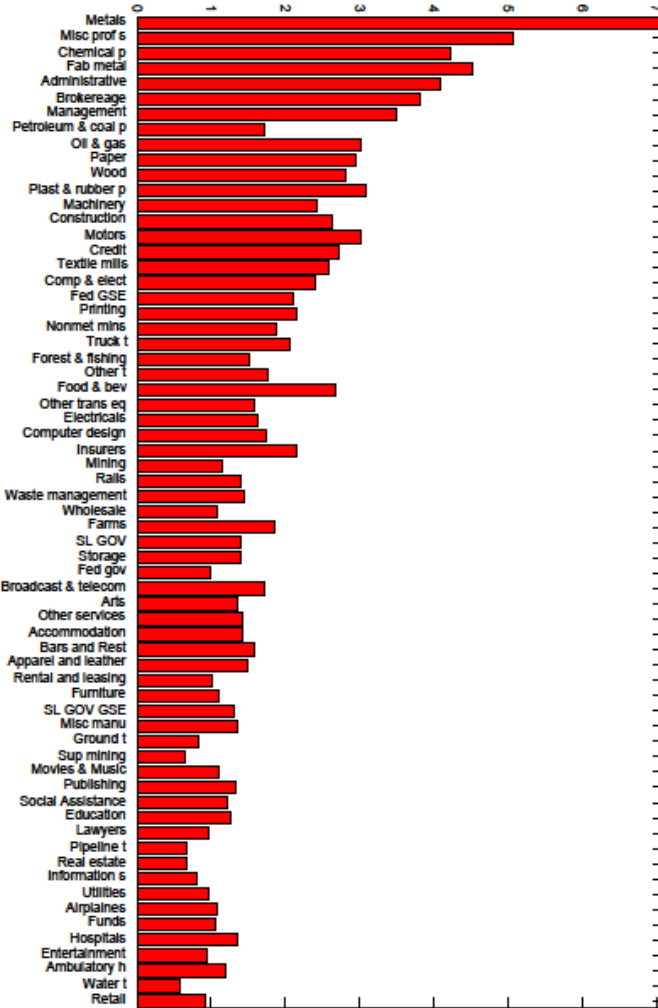
$$\tau_{i,t} = \left[ \frac{(c_{i,t}/p_{i,t})}{c_{i,1998}/p_{i,1998}} \right].$$

The lower panel of Figure ?? presents the average  $\tau_{i,t}$  across sectors for every period. One can see that at the peak of the crisis, the average firm in our model would have experienced a change in prices that would have made it act as if facing a 2% increase in intermediate input taxes. The standard deviation of this tax across sectors is about 1%. Thus, the amount of sectoral missallocation had an effect on the individual firm which would have been equivalent to the direct effect. Spill-over effects are roughly as big as the direct effects. Putting it differently, a firm in a given sector should be on average indifferent between removing its liquidity shocks or removing the liquidity shocks in the other sector of the economy. Now, such small taxes can have a large impact on output because, as argued in the public finance literature, the effects on intermediate taxes get compounded.

**Question 4. What sectors are the most affected by the aggregate implied liquidity shock?** Figure 5.1 answers this question. The sectors most affected are manufacturing sectors that provide intermediate inputs for final goods industries. These include metal products, chemical products, fabricated metal products, hydrocarbons and other industries related to the extraction and transformation of raw materials. Amongst the most sensible sectors we also find some service industries such as miscellaneous professional services and administrative, brokerage and management firms. Sectors that are least affected are retail, water transportation, and health related industries and entertainment. We don't have an answer for why this pattern. However, this pattern has to do more with the sectoral linkages than with the liquidity need of each sector. Figure 5.1 shows that sectors that are most sensible to the liquidity shock are in turn sectors that are suffering more from the misallocation. For example, the metal products is the most sensible sector because an aggregate 1% in liquidity shocks has the effects of an equivalent 7% increase in intermediate input taxes. We now turn the attention to answering questions about the relative importance of liquidity in the 3-digit sectors of the U.S. economy.





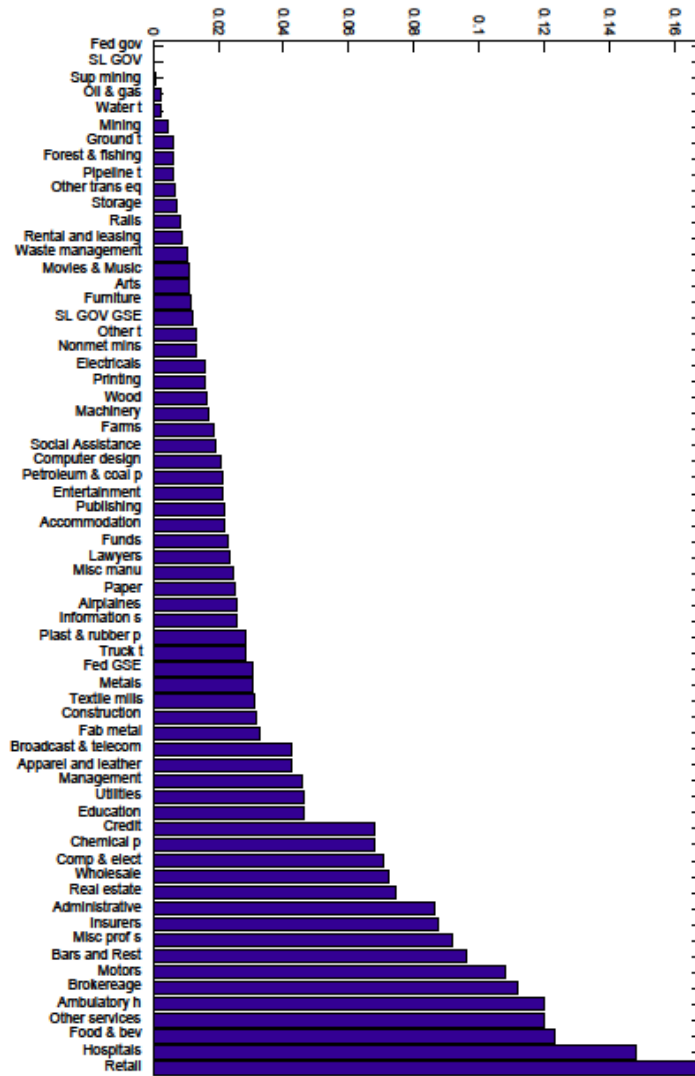


## 5.2 The Key Sectors

**Question 5.** Which sector is the most sensitive to its own liquidity shock? We now ask which sector responds most to a decrease in 1% in the liquidity of the same sector. Figure ?? shows the estimates of  $\frac{\partial y_i(\Theta_t)}{\chi_{i,t}}$  for every sector. The sectors with the highest internal sensitivity are roughly the same as the sector's that react the most to the aggregate liquidity shock but in this case, the ranking is lead by the motors (automotive) sector. Figure ?? hints

towards the explanation behind this ranking. It reports a scatter plot of the sectors impact on itself against the share of intermediate inputs and the share of the sector itself. Here the pattern is clear, the higher the share of intermediate inputs,  $\alpha_i$ , the higher the response to liquidity. Sectors with high  $\alpha_{ii}$  also respond more to liquidity. This last aspect is important because firms operating within the same sector should receive very similar liquidity shocks and if they are important supplying inputs for the sector itself, we should expect a large multiplier.

**Question 6. To what sector is the U.S. most sensitive?** Figure 5.2 reports the estimates of the response of aggregate GDP to the shock in one sector, i.e.,  $\langle \frac{\partial y(\theta_t)}{\partial x_{i,t}}, I \rangle$  for all  $i$ . The bars suggest that most important sector is *retail*. Other important sectors are industries such as motors and computer electronics, the most sensitive sector to themselves. This patterns points out to two sources of aggregate impact. The first force behind this ranking is the extent of sectoral linkages. For example, manufacturing industries are more interrelated than primary resource extraction industries, and this leads to larger aggregate effects. The second factor of influence is that sector's such as retail have important consumption shares. Hence, their aggregate impact is large simply because the sector is large in the computation of GDP although the sector is not dramatically affected as industrial sectors. Figure ?? presents the consumption shares in each sector, and the labor share of each sector against the sector's aggregate impact. Figure ?? shows that sector's with larger consumption shares lead to larger aggregate effects.



**Question 7. What sector causes the biggest spill overs?**

?? shows the average intermediate input tax rate induced on the rest of the sectors by every sector against the sector's aggregate impact. In this case, we find a low correlation between the aggregate impact rank and this spill-over rank. This suggests that the first order effect of a high intermediate shares or consumer shares are important in determining the sector's impact more than a possible misallocation. It also shows that shocks to sector's

such as retail will have a very large impact on the rest of the economy by affecting the demand for goods. Finally, Figure ?? describes the sector's that are affected most by a shock in a given other sector. The outcomes are natural.

### 5.3 Firm Level Analysis

**Question 8. What sector had the biggest impact?** We now try to go in more depth towards the sector's that could have been key during the recession. We limit the analysis to 9 manufacturing industries and construction.<sup>18</sup> We gather data from COMPUSTAT for all the industries and compute the ratio of costs over sales for each sector. We corroborated a negative relationship with this measure and sales growth (in agreement with the countercyclical markups). This feature is in line with our model. We also corroborated a negative relationship with our measure of illiquidity and an increasing working capital over costs share. The implied reduction in GDP is shown in Figure ?. The figure shows that these sector's implied liquidity drop could explain roughly 1/3 of the drop in GDP purely from missallocation effects. That is, keeping hours constant, prices would have to fall in this hypothetical world by roughly 0.5%. Figure ? presents a decomposition of the drop i GDP by the contribution of each sector. Roughly all sector's play a similar role except for wood products and construction that have a lower impact. This result is surprising given that construction fell dramatically during the crises. Our result could be affected because of our imperfect measure of liquidity or a size bias on the COMPUSTAT sample.

**Question 9. Which one caused the biggest spill overs?**

Figure ? presents a measurement of the sector's induced taxes during the recession as a measure of the spill-over effects induced by each sector. At the top of the rank are leather and apparel as well as printing.

### 5.4 Failure of the Model

It is important to discuss what features is this model missing. We note several things. [1] The model lacks any form of price rigidities. In particular, wage rigidities could possibly explain the much sharper reduction in hours as suggested by Figure ?. We see in that hours fall more in the data compared to the model. If anything, a rigidity of the sort would increase the multiplier that we calculated in the first part. [2] A second issue is related to the fact that we are neglecting any form of durability in final goods. Consumer durables would potentially amplify the output of industries in the durables sectors by causing

---

<sup>18</sup>Construction, Wood Products, Non-metallic Mineral Products, Metallic Products, Electrical and Computer Products, Furniture, Food and Beverage, Apparel and Leather and Printing.

changes in relative demand. In our model, consumption shares are constant fractions of labor income. [3] A third concern is that we are allowing a high degree of production elasticity substitution. In the model, sectors whose suppliers are affected by reductions in liquidity easily substitute production inputs with other inputs or labor. The model may be missing a larger amount of rigidities in the production process. [4] A final concern is that we are treating sectors in a symmetric way. This is a problem because sectors such as retail, wholesale and warehousing and transportation are sectors that don't transform products in a the same way manufacturing industries. Often these are simple intermediaries charging a constant markup so the decreasing returns to scale assumption may not be appropriate.

## 6 Conclusion

This paper argues that the network of production links in an economy can matter substantially for the transmission of financial shocks. To make this point, we formulated an economy in the most simple way possible. We introduced financial shocks that are orthogonal across sectors. That is, we formulated a model such that the liquidity shock in one sector has no effects on the intermediate inputs wedge of other sectors, although there are spill-over effects via the effects on prices that cause inefficient reallocation.

We provided several analytic examples of liquidity shocks to analyze their propagation in particular network structures. We then took the structure of the U.S. I-O and calibrated our model. We asked what is the liquidity multiplier in the U.S. Our experiment showed a multiplier of 3.5. compared with a horizontal economy whose multiplier is 1.

We believe the exercise illustrates the that the effects of liquidity shocks can be quite dramatic if production is organized with industrial linkages. We speculate that if one were to introduce the possibility of demand changes via durable consumption preferences, nominal rigidities, or low short-run input substitution, the effects could be even more dramatic. This extensions can be studied bringing this framework into richer environments. [Literature here]

There are other questions that are relevant for the theoretical study of financial frictions in networks. Early work of Kiyotaki and Moore [Credit Chains, Moore] noted that disruptions in the payments chain have important welfare implications. We abstracted from any stratetigic/time dimension form of trade credit because in our model, trade credit moves jointly with liquidity. However, we should study a model which can explain disruptions in the supply chain. In our model, the net-work structure is exogenous. There is a growing literature on endogenous network formation [Oberfeldt]. Part of this network formation has to do with technological processes but a lot can be do to financial innovation in a way that

can give us a better understanding of the theory of the firm.

Optimal allocation of liquidity?

## 7 Appendix A

**firm's problem** We impose that firms face trade credit contracts are subject to a limited enforcement problem. These depend on an amount of liquidity that characterize the firm's problem. We begin describing the firm's problem. Firm  $i$  maximizes profits

$$\Pi_i = \max_{\sigma_i, x_i} p_i y_i - c_i x_i$$

subject to

$$y_i = z_i x_i^{\alpha_i}$$

$$(1 - \sigma_i) c_i x_i \leq w_i \tag{20}$$

$$(1 - \theta_i) p_i y_i \leq p_i y_i - \sigma_i c_i x_i. \tag{21}$$

In the expression above  $c_i$  is the marginal cost of  $x_i$  which by the assumption that  $\sum_{j \in N_i} \alpha_{ij} = 1$ , is constant and independent of  $x_i$ . The first constraint is the technological constraint of the firm. The second constraint states that a fraction  $(1 - \sigma_i)$ , (chosen by the firm) has to be paid up-front with liquid funds  $w_i$ . Liquid funds are given by an exogenously. Changes in the availability of liquid funds are the focus of interest in this paper. In addition we assume, following the contracting literature, that firm's can pledge at most  $\theta_i$  of their output to pay their suppliers. Thus,  $\sigma_i c_i x_i$  is the amount of trade credit obtained from suppliers. If they choose to default on suppliers, they loose the fraction  $\theta_i$  of the firm's income. Hence, upon default, the firm keeps the fraction  $(1 - \theta_i) p_i y_i$ . Thus, the third constraint is an incentive constraint that states that the fraction of output they get to keep should they choose to default on suppliers must exceed the revenue firm's expect to make minus the amount it owes after it pays for part of its inputs with liquid funds. By rearranging this constraint, we obtain an equivalent constraint,  $\sigma_i c_i x_i \leq \theta_i p_i y_i$ . This one reads that the amount that the firm can owe to its suppliers after it payed from a certain fraction in advance must not exceed the pledgeable amount of output,  $\theta_i p_i y_i$ . In the specific examples that follow, we let the liquid funds to be some proportion of the firms output  $w_i = \omega_i p_i y_i$ .

**Lemma 5.** *The Liquidity and Enforcement constraints (??) and ??) bind jointly if and only if*

$$\theta_i + \omega_i < \alpha_i$$

*The firm's problem is characterized by the following first order condition:*

$$c_i x_i = \phi_i p_i y_i$$



where  $\phi_i = \min \{ \alpha_i, (\theta_i + \omega_i) \}$ .

[NEED TO REDO PROOF]

## 8 Appendix B: Proofs

**Proof of Proposition 1** *Vertical Economy.* Let us first consider the Vertical economy. Firm 1 maximizes the following objective function.

$$\max p_1 A_1 n_{v1}^{\alpha_1} - w n_1$$

This yields FOC

$$p_1 \alpha_1 \frac{y_1}{n_1} = w$$

Firm 2 maximizes the following objective function.

$$\max p_2 A_2 n_{v2}^{\alpha_2} y_{v1}^{\beta_2} - w n_2 - p_1 y_1$$

This yields FOCs

$$\alpha_2 p_2 \frac{y_2}{n_2} = w \quad \text{and} \quad \beta_2 p_2 \frac{y_2}{y_1} = p_1$$

Firm 2's expenditure on goods is given by

$$w n_2 + p_1 y_1 = \alpha_2 p_2 y_2 + \beta_2 p_2 y_2 = (\alpha_2 + \beta_2) p_2 y_2$$

Firm 3 solves a similar problem to that of firm 2; it's FOC's are given by

$$\alpha_3 P \frac{y_3}{n_3} = w \quad \text{and} \quad \beta_3 P \frac{y_3}{y_2} = p_2$$

Thus, combining () with () and (), we get that.

$$\begin{aligned} \alpha_3 \frac{Y}{n_3} &= \frac{w}{P} \\ \alpha_2 \beta_3 \frac{Y}{n_2} &= \frac{w}{P} \\ \alpha_1 \beta_2 \beta_3 \frac{Y}{n_1} &= \frac{w}{P} \end{aligned}$$

Finally, the optimality condition for the household is given by

$$\frac{v'(N)}{u'(C)} = \frac{w}{P}$$

Combining this with (1)-(3) yields the conditions for the

*Horizontal Economy.* In the horizontal economy, all firms are identical and maximize the following objective.

$$\max p_i A_i n_i^{\alpha_i} - w n_i$$

the FOC's for each firm is given by

$$p_i \alpha_i \frac{y_i}{n_i} = w$$

Finally the household maximizes the following objective

$$\max P y_1^{\beta_1} y_2^{\beta_2} y_3^{\beta_3} - p_1 y_1 - p_2 y_2 - p_3 y_3$$

It's FOC is given by

$$\begin{aligned} \beta_2 \beta_3 P \frac{Y}{y_1} &= p_1 \\ \beta_3 P \frac{Y}{y_2} &= p_2 \\ P \frac{Y}{y_3} &= p_3 \end{aligned}$$

Combining, we get the following conditions, which are exactly the same as in the vertical economy.

$$\begin{aligned} \alpha_1 \beta_2 \beta_3 \frac{Y}{n_1} &= \frac{w}{P} \\ \alpha_2 \beta_3 \frac{Y}{n_2} &= \frac{w}{P} \\ \alpha_3 \frac{Y}{n_3} &= \frac{w}{P} \end{aligned}$$

The household's optimality condition remains the same as (4). Thus, combining this with (1)-(3) yields the same conditions

**Proof of Proposition 2** *Vertical Economy.* Let's first consider the vertical economy with collateral constraints. Firm 1 maximizes the following objective

$$\max p_1 y_1 - w n_1$$

subject to

$$\begin{aligned} y_1 &= A_1 n_{v1}^{\alpha_1} \\ w n_1 &\leq \chi_1 p_1 y_1 \end{aligned}$$

If collateral constraint is binding then

$$w n_1 = \chi_1 p_1 y_1$$

otherwise the firm chooses labor according to it's unconstrained FOC given by

$$p_1 \alpha_1 \frac{y_1}{n_1} = w$$

We may summarize this in the following way.

$$w n_1 = \phi_1 \alpha_1 p_1 y_1$$

where  $\phi_i = \min \left\{ 1, \frac{\chi_{hi}}{\alpha_i} \right\}$ .

Now consider the problem of firm 2:

$$\max p_2 y_2 - w n_2 - p_1 y_1$$

subject to

$$\begin{aligned} y_2 &= A_2 n_{v2}^{\alpha_2} y_{v1}^{\beta_2} \\ w n_2 + p_1 y_1 &\leq \chi_2 p_2 y_2 \end{aligned}$$

Firm 2's cost minimization is given by

$$\min w n_2 + p_1 y_1$$

subject to

$$y_2 = A_2 n_2^{\alpha_2} y_1^{\beta_2}$$

This implies that the firm's optimal choices for inputs must satisfy

$$\frac{1}{\alpha_2}wn_2 = \frac{1}{\beta_2}p_1y_1$$

Firm 2's unconstrained expenditure on goods would be given by

$$wn_2 + p_1y_1 = (\alpha_2 + \beta_2)p_2y_2$$

Thus firm 2 is constrained iff

$$\alpha_2 + \beta_2 > \chi_2$$

If constrained, then

$$\begin{aligned} wn_2 + wn_2\frac{\beta_2}{\alpha_2} &= \chi_2p_2y_2 \\ wn_2\left(\frac{\alpha_2 + \beta_2}{\alpha_2}\right) &= \chi_2p_2y_2 \end{aligned}$$

If unconstrained, then

$$\begin{aligned} wn_2\left(\frac{\alpha_2 + \beta_2}{\alpha_2}\right) &= (\alpha_2 + \beta_2)p_2y_2 \\ wn_2 &= \alpha_2p_2y_2 \end{aligned}$$

Thus,

$$wn_2 = \phi_2\alpha_2p_2y_2$$

where  $\phi_2 = \min\left\{1, \frac{\chi_2}{\alpha_2 + \beta_2}\right\}$ .

Similarly for firm 3 we have that

$$\frac{1}{\alpha_3}wn_3 = \frac{1}{\beta_3}p_2y_2$$

Thus

$$wn_3 = \phi_3\alpha_3p_2y_2$$

where  $\phi_3 = \min\left\{1, \frac{\chi_3}{\alpha_3 + \beta_3}\right\}$ .

Plugging this into firm 2's problem

$$\begin{aligned}
 p_2 y_2 &= wn_3 \frac{\beta_3}{\alpha_3} \\
 &= \frac{\beta_3}{\alpha_3} \alpha_3 \frac{\chi_3}{\alpha_3 + \beta_3} PY \\
 &= \chi_3 \frac{\beta_3}{\alpha_3 + \beta_3} PY
 \end{aligned}$$

Therefore, for firm 3 we have

$$wn_3 = \phi_3 \alpha_3 PY$$

which corresponds with equation (). For firm 2 we have

$$wn_2 = \phi_2 \alpha_2 p_2 y_2$$

where

$$p_2 y_2 = \frac{\beta_3}{\alpha_3} wn_3 = \beta_3 \phi_3 PY$$

This implies

$$wn_2 = \phi_2 \phi_3 \alpha_2 \beta_3 PY$$

which corresponds with equation (). And finally for firm 1 we have

$$wn_1 = \phi_1 \alpha_1 p_1 y_1$$

where

$$p_1 y_1 = \frac{\beta_2}{\alpha_2} wn_2 = \frac{\beta_2}{\alpha_2} \phi_2 \phi_3 \alpha_2 \beta_3 PY$$

This implies

$$wn_1 = \phi_1 \phi_2 \phi_3 \alpha_1 \beta_2 \beta_3 PY$$

which corresponds with equation ().

*Horizontal Economy.* Again, each firm solves an identical problem. Firm  $i$  chooses  $n_i$  to maximize

$$\max p_i A_i n_i^{\alpha_i} - wn_i$$

subject to

$$wn_i \leq \chi_i p_i y_i$$

thus, if it is binding then

$$\chi_i p_i y_i = wn_i$$

otherwise

$$p_i \alpha_i \frac{y_i}{n_i} = w$$

Therefore,

$$\phi_i \alpha_i p_i y_i = w n_i$$

where  $\phi_i = \min \left\{ 1, \frac{\chi_{hi}}{\alpha_i} \right\}$ ,  $\forall i$ .

Finally the household maximizes the following objective

$$\max P y_1^{\beta_2 \beta_3} y_2^{\beta_3} y_3 - p_1 y_1 - p_2 y_2 - p_3 y_3$$

It's FOC is given by

$$\begin{aligned} \beta_2 \beta_3 P \frac{Y}{y_1} &= p_1 \\ \beta_3 P \frac{Y}{y_2} &= p_2 \\ P \frac{Y}{y_3} &= p_3 \end{aligned}$$

Combining these with () yields

$$\begin{aligned} \phi_1 \alpha_1 \beta_2 \beta_3 P Y &= w n_1 \\ \phi_2 \alpha_2 \beta_3 P Y &= w n_2 \\ \phi_3 \alpha_3 P Y &= w n_3 \end{aligned}$$

which corresponds with equations ()-().

**Proof of Proposition 3** In the horizontal economy, we have that

$$\begin{aligned} n_{h3} &= (\phi_{h3}) \alpha_3 \frac{Y}{V'(N) / U'(C)} \\ n_{h2} &= (\phi_{h2}) \alpha_2 \beta_3 \frac{Y}{V'(N) / U'(C)} \\ n_{h1} &= (\phi_{h1}) \alpha_1 \beta_2 \beta_3 \frac{Y}{V'(N) / U'(C)} \end{aligned}$$

thus

$$\frac{Y}{N} = \frac{1}{\alpha_3 (\phi_{h3}) + \alpha_2 \beta_3 (\phi_{h2}) + \alpha_1 \beta_2 \beta_3 (\phi_{h1})} \frac{V'(N)}{U'(C)}$$

From this, we can again back out the aggregate labor wedge  $1 - \tau$  defined by ().

In the vertical economy, we have that

$$\begin{aligned} n_{h3} &= (\phi_{h3}) \alpha_3 \frac{Y}{W/P} \\ n_{h2} &= (\phi_{v2} \phi_{v3}) \alpha_2 \beta_3 \frac{Y}{W/P} \\ n_{h1} &= (\phi_{v1} \phi_{v2} \phi_{v3}) \alpha_1 \beta_2 \beta_3 \frac{Y}{W/P} \end{aligned}$$

thus

$$\frac{Y}{N} = \frac{1}{\alpha_3 (\phi_{v3}) + \alpha_2 \beta_3 (\phi_{v2} \phi_{v3}) + \alpha_1 \beta_2 \beta_3 (\phi_{v1} \phi_{v2} \phi_{v3})} \frac{W}{P}$$

From this, we can again back out the aggregate labor wedge  $1 - \tau$  defined by (). QED.

**Proof of Proposition 4** [NEED TO FIX ALPHAS AND BETAS]

In the horizontal economy

$$M_h = w (n_1 + n_2 + n_3)$$

In the vertical economy

$$M_v = w \left[ n_{v1} + \left( 1 + \frac{\alpha_2}{\beta_2} \right) n_{v2} + \left( 1 + \frac{\alpha_3}{\beta_3} \right) n_{v3} \right]$$

We can translate that into real terms

(i) the liquidity needed in the vertical economy to acheive  $Y_v = Y$  is given by

$$\Phi_v = Y \left[ \phi_{v3} + \phi_{v2} \alpha_3 \left( \frac{\phi_{v3}}{\beta_3 + \alpha_3} \right) + \phi_{v1} \alpha_2 \alpha_3 \left( \frac{\phi_{v2}}{\beta_2 + \alpha_2} \right) \left( \frac{\phi_{v3}}{\beta_3 + \alpha_3} \right) \right]$$

(i) the liquidity needed in the horizontal economy to acheive  $Y_h = Y$  is given by

$$\Phi_h = Y [\phi_{h3} + \phi_{h2} \alpha_3 + \phi_{h1} \alpha_2 \alpha_3]$$

**Proof of Proposition ??** Given our specification for utility in (), we have that

$$\frac{V'(N)}{U'(Y)} = Y$$

*Frictionless Economy.* Thus, in the economy without frictions, we have that

$$\alpha_3 \frac{Y_\varepsilon}{n_{\varepsilon 3}} = \alpha_2 \beta_3 \frac{Y_\varepsilon}{n_{\varepsilon 2}} = \alpha_1 \beta_2 \beta_3 \frac{Y_\varepsilon}{n_{\varepsilon 1}} = \frac{V'(N)}{U'(Y)}$$

Thus,

$$\begin{aligned}n_3 &= \alpha_3 \\n_2 &= \alpha_2\beta_3 \\n_1 &= \alpha_1\beta_2\beta_3\end{aligned}$$

Substituting these values for  $n$  into () gives us our expression for aggregate output ().

*Horizontal Economy.* In the horizontal economy, the unique equilibrium allocation is given by

$$(\phi_{h3})\alpha_3\frac{Y_h}{n_{h3}} = (\phi_{h2})\alpha_2\beta_3\frac{Y_h}{n_{h2}} = (\phi_{h1})\alpha_1\beta_2\beta_3\frac{Y_h}{n_{h1}} = Y_h$$

This implies

$$\begin{aligned}n_{h3} &= \phi_{h3}\alpha_3 \\n_{h2} &= \phi_{h2}\alpha_2\beta_3 \\n_{h1} &= \phi_{h1}\alpha_1\beta_2\beta_3\end{aligned}$$

Thus, there are no spill over effects other than income-substitution effects coming from the real wage. Therefore, output is given by

$$Y_h =$$

which corresponds with equation ().

*Vertical Economy.* In the vertical economy, the unique equilibrium allocation is given by

$$(\phi_{v3})\alpha_3\frac{Y_v}{n_{v3}} = (\phi_{v2}\phi_{v3})\alpha_2\beta_3\frac{Y_v}{n_{v2}} = (\phi_{v1}\phi_{v2}\phi_{v3})\alpha_1\beta_2\beta_3\frac{Y_v}{n_{v1}} = Y_v$$

This implies

$$\begin{aligned}n_{v3} &= \phi_{v3}\alpha_3 \\n_{v2} &= \phi_{v2}\phi_{v3}\alpha_2\beta_3 \\n_{v1} &= \phi_{v1}\phi_{v2}\phi_{v3}\alpha_1\beta_2\beta_3\end{aligned}$$

Aggregate output is given by

$$Y_v =$$



which corresponds with equation (). QED

**Proof of Proposition 5** These expressions follow from taking the derivative of ().

**Proof of Proposition 6** Now, let's think of a fall in aggregate liquidity. We drop  $\phi$  down by  $x$ -percent.

In the horizontal economy.

$$\begin{aligned}\frac{d \log Y_h}{d \log \phi} &= \frac{d \log Y_h}{d \log \phi_{h1}} + \frac{d \log Y_h}{d \log \phi_{h2}} + \frac{d \log Y_h}{d \log \phi_{h3}} \\ &= \alpha_1 \beta_2 \beta_3 + \alpha_2 \beta_3 + \alpha_3\end{aligned}$$

In the vertical economy

$$\begin{aligned}\frac{d \log Y_v}{d \log \phi} &= 3 \frac{d \log Y_h}{d \log \phi_{h1}} + 2 \frac{d \log Y_h}{d \log \phi_{h2}} + \frac{d \log Y_h}{d \log \phi_{h3}} \\ &= 3\alpha_1 \beta_2 \beta_3 + 2\alpha_2 \beta_3 + \alpha_3\end{aligned}$$

It follows that  $\frac{d \log Y_v}{d \log \phi} > \frac{d \log Y_h}{d \log \phi}$ .

**Proof of Proposition 7** Follows directly from equations ()-(). QED.

**Proof of Proposition 7** In order to answer this question, we need only consider a tightening of the collateral constraint of firm 2. First, consider the effects of employment from a tightening of the collateral constraint of firm 2 implies that it will buy both less labor and less intermediate inputs

$$n_{v2} = \phi_{v2} \phi_{v3} \alpha_2 \beta_3$$

thus,

$$\frac{d \log n_{v2}}{d \log \phi_{v2}} = 1$$

labor of firm 1 is given by  $n_{v1} = \phi_{v1} \phi_{v2} \phi_{v3} \alpha_1 \beta_2 \beta_3$ , which implies that

$$\frac{d \log n_{v1}}{d \log \phi_{v2}} = 1$$

and labor of firm 3 is unchanged.

$$\frac{d \log n_{v3}}{d \log \phi_{v2}} = 0$$

Now, let's look at output. Output of firm 1  $y_1 = A_1 n_{v1}^{\alpha_1}$  decreases

$$\log y_{v1} = \log A_1 + \alpha_1 \log n_{v1}$$

thus

$$\frac{d \log y_{v1}}{d \log \phi_{v2}} = \alpha_1 \frac{d \log n_{v1}}{d \log \phi_{v2}} = \alpha_1$$

output of firm 2  $y_{v2} = A_2 n_{v2}^{\alpha_2} y_{v1}^{\beta_2}$

$$\frac{d \log y_{v2}}{d \log \phi_{v2}} = \alpha_2 \frac{d \log n_{v2}}{d \log \phi_{v2}} + \beta_2 \frac{d \log y_{v1}}{d \log \phi_{v2}} = \alpha_2 + \beta_2 \alpha_1$$

output of firm 3  $y_{v3} = A_3 n_{v3}^{\alpha_3} y_{v2}^{\beta_3}$

$$\begin{aligned} \frac{d \log y_{v3}}{d \log \phi_{v2}} &= \alpha_3 \frac{d \log n_{v3}}{d \log \phi_{v2}} + \beta_3 \frac{d \log y_{v2}}{d \log \phi_{v2}} \\ &= \alpha_2 \beta_3 + \alpha_1 \beta_2 \beta_3 \end{aligned}$$

which corresponds with the fall in aggregate output in ().

Finally, let's look at prices. What happens to the real wage?

$$\frac{w}{p} = \frac{V'(N)}{U'(Y)} = Y$$

Thus the effect on the real wage is given by

$$\frac{d \log w}{d \log \phi_{v2}} = \frac{d \log Y}{d \log \phi_{v2}} = \alpha_2 \beta_3 + \alpha_1 \beta_2 \beta_3$$

Thus, for firm 2 we have that

$$w n_2 = \phi_2 \alpha_2 p_2 y_2$$

therefore

$$p_2 = \frac{w n_2}{\alpha_2 \phi_2 y_2}$$

thus

$$\begin{aligned} \frac{d \log p_2}{d \log \phi_{v2}} &= \frac{d \log w}{d \log \phi_{v2}} + \frac{d \log n_2}{d \log \phi_{v2}} - \frac{d \log \phi_{v2}}{d \log \phi_{v2}} - \frac{d \log y_2}{d \log \phi_{v2}} \\ &= \beta_3 (\alpha_2 + \alpha_1 \beta_2) + 1 - 1 - (\alpha_2 + \beta_2 \alpha_1) \\ &= -(1 - \beta_3) (\alpha_2 + \alpha_1 \beta_2) < 0 \end{aligned}$$

therefore, the price of firm 2 goes up. Also expenditure of firm 3 on intermediate goods  $p_2y_2$  goes down

$$\frac{d \log p_2}{d \log \phi_{v2}} + \frac{d \log y_2}{d \log \phi_{v2}} = \frac{d \log w}{d \log \phi_{v2}} = \beta_3 (\alpha_2 + \alpha_1 \beta_2) < 0$$

thus price of firm 2 goes up.

price of firm 1. Can pin it down either from this

$$wn_1 = \phi_1 \alpha_1 p_1 y_1$$

or from this.

$$\frac{1}{\alpha_2} wn_2 = \frac{1}{\beta_2} p_1 y_1$$

thus

$$\begin{aligned} \frac{d \log p_1}{d \log \phi_{v2}} &= \frac{d \log w}{d \log \phi_{v2}} + \frac{d \log n_1}{d \log \phi_{v2}} - \frac{d \log y_1}{d \log \phi_{v2}} \\ &= \beta_3 (\alpha_2 + \alpha_1 \beta_2) + 1 - \alpha_1 > 0 \end{aligned}$$

or

$$\begin{aligned} \frac{d \log p_1}{d \log \phi_{v2}} &= \frac{d \log w}{d \log \phi_{v2}} + \frac{d \log n_2}{d \log \phi_{v2}} - \frac{d \log y_1}{d \log \phi_{v2}} \\ &= \beta_3 (\alpha_2 + \alpha_1 \beta_2) + 1 - \alpha_1 > 0 \end{aligned}$$

price of firm 1 goes down after collateral constraint tightens, since there is less demand for this good.

suppose there is a collateral shock to firm 1. just to understand,

$$wn_2 = \phi_2 \alpha_2 p_2 y_2$$

thus where

$$y_2 = A_2 n_{v2}^{\alpha_2} y_{v1}^{\beta_2}$$

so that

$$\begin{aligned} \frac{d \log y_2}{d \log \phi_{v1}} &= \beta_2 \frac{d \log y_1}{d \log \phi_{v1}} \\ &= \beta_2 \alpha_1 \end{aligned}$$

$$\begin{aligned}
p_2 &= \frac{d \log w}{d \log \phi_{v1}} + \frac{d \log n_2}{d \log \phi_{v1}} - \frac{d \log y_2}{d \log \phi_{v1}} \\
&= \frac{d \log Y}{d \log \phi_{v1}} + 1 - \frac{d \log y_2}{d \log \phi_{v1}} \\
&= \alpha_3 + 1 - \beta_2 \alpha_1 > 0
\end{aligned}$$

SOMETHING IS WRONG HERE—SEE MATLAB CODE

price of firm 2 goes down after collateral constraint 1 tightens, since inputs are now more expensive.

**Proof of Proposition 8** In the horizontal economy

$$\begin{aligned}
(\phi_{h3}) \alpha_3 \frac{Y_h}{n_{h3}} &= V'(N_h) / U'(Y_h) \\
(\phi_{h2}) \alpha_2 \beta_3 \frac{Y_h}{n_{h2}} &= V'(N_h) / U'(Y_h) \\
(\phi_{h1}) \alpha_1 \beta_2 \beta_3 \frac{Y_h}{n_{h1}} &= V'(N_h) / U'(Y_h)
\end{aligned}$$

with power utility, linear disutility of labor, we have

$$\begin{aligned}
n_{h3} &= \phi_{h3} \alpha_3 \\
n_{h2} &= \phi_{h2} \alpha_2 \beta_3 \\
n_{h1} &= \phi_{h1} \alpha_1 \beta_2 \beta_3
\end{aligned}$$

Thus,

$$N = n_{h1} + n_{h2} + n_{h3}$$

In the vertical economy

$$\begin{aligned}
(\phi_{v3}) \alpha_3 \frac{Y_v}{n_{v3}} &= V'(N_v) / U'(Y_v) \\
(\phi_{v2} \phi_{v3}) \alpha_2 \beta_3 \frac{Y_v}{n_{v2}} &= V'(N_v) / U'(Y_v) \\
(\phi_{v1} \phi_{v2} \phi_{v3}) \alpha_1 \beta_2 \beta_3 \frac{Y_v}{n_{v1}} &= V'(N_v) / U'(Y_v) \\
N_v &= n_{v1} + n_{v2} + n_{v3}
\end{aligned}$$

with power utility, linear

$$\begin{aligned} n_{v3} &= \phi_{v3} \alpha_3 \\ n_{v2} &= \phi_{v2} \phi_{v3} \alpha_2 \beta_3 \\ n_{v1} &= \phi_{v1} \phi_{v2} \phi_{v3} \alpha_1 \beta_2 \beta_3 \end{aligned}$$

Can we get TFP?

### Proof of Lemma Composite cost

*Proof.* The fact that  $\sum_{j \in I_i} \alpha_{ij} = 1$  implies that problem is homogeneous and hence  $c_i$  is independent of  $x_i$ . The solution to this cost minimization problem is obtained by forming a Lagrangean. Let  $\lambda_i$  be the Lagrangean associated with the capacity constraint. Then we obtain the following first order condition:

$$p_j x_{ij} = \lambda_i \alpha_{ij} \prod_{j \in I_i} x_{ji}^{\alpha_{ij}} = \lambda_i \alpha_{ij} x_i, \quad j \in N$$

With this, we know that the expense share firm  $i$ 's use of  $j$ 's input is proportional  $\alpha_{ij}$

$$\frac{p_j x_{ij}}{p_k x_{ik}} = \frac{\alpha_{ij}}{\alpha_{ik}} \quad (22)$$

or alternatively:

$$p_j x_{ij} = \alpha_{ij} c_i x_i$$

This result implies hence,

$$\begin{aligned} x_i &= c_i^{\sum_{j \in N_i} \alpha_{ij}} \cdot \prod_{j \in N_i} \left( \frac{\alpha_{ij}}{p_j} \right)^{\alpha_{ij}} \cdot \prod_{j \in N_i} x_{ji}^{\alpha_{ij}} \rightarrow \\ c_i &= \prod_{j \in N_i} \left( \frac{p_j}{\alpha_{ij}} \right)^{\alpha_{ij}} \end{aligned}$$

We are in time to describe the household's problem. □

**Proof of Proposition 9**

$$\begin{aligned}
\left(\frac{1}{n}(I - (1 - \alpha)W')^{-1}\mathbf{1}\right)' \mathbf{1}\alpha \log h &= v'(\mathbf{z} + \boldsymbol{\phi}) + \text{const} + \left(\frac{1}{n}(I - (1 - \alpha)W')^{-1}\mathbf{1}\right)' (I - (1 - \alpha)W)\mathbf{p} \\
\frac{1}{n}\left((I - (1 - \alpha)W')^{-1}\mathbf{1}\right)' \mathbf{1}\alpha \log h &= v'(\mathbf{z} + \boldsymbol{\phi}) + \text{const} + \frac{1}{n}\mathbf{1}'(I - (1 - \alpha)W')^{-1}(I - (1 - \alpha)W)\mathbf{p} \\
\frac{1}{n}\left((I - (1 - \alpha)W')^{-1}\mathbf{1}\right)' \mathbf{1}\alpha \log h &= v'(\mathbf{z} + \boldsymbol{\phi}) + \text{const} + \frac{1}{n}\mathbf{1}'(I - (1 - \alpha)W)^{-1}(I - (1 - \alpha)W)\mathbf{p} \\
\frac{1}{n}\left((I - (1 - \alpha)W')^{-1}\mathbf{1}\right)' \mathbf{1}\alpha \log h &= v'(\mathbf{z} + \boldsymbol{\phi}) + \text{const} + \frac{1}{n}\mathbf{1}'\mathbf{p} \\
\left(\frac{1}{n}(I - (1 - \alpha)W')^{-1}\mathbf{1}\right)' \mathbf{1}\alpha \log h &= v'(\mathbf{z} + \boldsymbol{\phi}) + \text{const} + \frac{1}{n}\mathbf{1}'\mathbf{p}
\end{aligned}$$

**Proof of Lemma Sales** Acemoglu et al. show that

$$s' = (h/n)\mathbf{1}'(I - (1 - \alpha)W')^{-1} = hv'$$

whereas with financial frictions we have that

$$s' = (h/n)\mathbf{1}'(I - (1 - \alpha)W'\Phi')^{-1}$$

We'll first show the Acemoglu result then show how this doesn't apply in our economy. The market clearing condition for commodity  $i$  is given by

$$c_i + \sum_{j=1}^n x_{ji} = x_i$$

Multiplying this by  $p_i$  we have that

$$p_i c_i + p_i \sum_{j=1}^n x_{ji} = p_i x_i$$

Plugging in consumption levels and firms' input demands, we have that

$$\frac{h}{n} + (1 - \alpha) \sum_{j=1}^n w_{ji} p_j x_j = p_i x_i$$

Letting  $s_i = p_i x_i$  denote sales, we have that

$$s_i = \frac{h}{n} + (1 - \alpha) \sum_{j=1}^n w_{ji} s_j$$

stacking, we have that

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \mathbf{1} \frac{h}{n} + (1 - \alpha) \begin{bmatrix} w_{11} & w_{21} & \cdots & w_{n1} \\ w_{12} & w_{22} & \cdots & w_{n2} \\ \vdots & & & \\ w_{1n} & & & w_{nn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

we can thus write

$$\mathbf{s} = \mathbf{1} \frac{h}{n} + (1 - \alpha) W' \mathbf{s}$$

where

$$\mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

taking the transpose of this, we have that

$$\mathbf{s}' = \mathbf{1}' \frac{h}{n} + (1 - \alpha) \mathbf{s}' W$$

rearranging, we have that

$$\mathbf{s}' = \frac{h}{n} \mathbf{1}' (I - (1 - \alpha) W)^{-1} = h v'$$

In our paper, when we plug in firm optimality conditions ( ) into ( ), we get that

$$\frac{h}{n} + (1 - \alpha) \sum_{j=1}^n \phi_j w_{ji} p_j x_j = p_i x_i$$

Thus

$$s_i = \frac{h}{n} + (1 - \alpha) \sum_{j=1}^n \phi_j w_{ji} s_j$$

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \mathbf{1} \frac{h}{n} + (1 - \alpha) \begin{bmatrix} \phi_1 w_{11} & \phi_2 w_{21} & \cdots & \phi_n w_{n1} \\ \phi_1 w_{12} & \phi_2 w_{22} & \cdots & \phi_n w_{n2} \\ \vdots & & & \\ \phi_1 w_{1n} & & & \phi_n w_{nn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

we may rewrite this as

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \mathbf{1} \frac{h}{n} + (1 - \alpha) \begin{bmatrix} w_{11} & w_{21} & \cdots & w_{n1} \\ w_{12} & w_{22} & \cdots & w_{n2} \\ \vdots & & & \\ w_{1n} & & & w_{nn} \end{bmatrix} \begin{bmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & \phi_n \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

Therefore

$$\mathbf{s} = \mathbf{1} \frac{h}{n} + (1 - \alpha) W' \Phi \mathbf{s}$$

where

$$\Phi = \begin{bmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & \phi_n \end{bmatrix}$$

taking the transpose of this, we have that

$$\mathbf{s}' = \mathbf{1}' \frac{h}{n} + (1 - \alpha) \mathbf{s}' \Phi' W$$

rearranging, we have that

$$\mathbf{s}' = \frac{h}{n} \mathbf{1}' (I - (1 - \alpha) \Phi' W)^{-1}$$