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Abstract

Larger firms (by sales or employment) have higher leverage. This pattern is explained using a model in which firms produce multiple varieties, acquire new varieties from their inventors, and borrow against the future cash flow of the firm with the option to default. A variety can die with a constant probability, implying that firms with more varieties (bigger firms) have a lower variance of sales growth and, in equilibrium, higher leverage. In this setup, a drop in the risk-free rate increases the value of an acquisition more for bigger firms because of their higher leverage: they can (and do) borrow a larger fraction of their future cash flow. The drop causes existing firms to buy more of the new varieties arriving into the economy, resulting in a lower startup rate and greater concentration of sales.

Keywords: Startup rates, concentration, leverage, firm dynamics

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1 Introduction

Across a range of advanced economies, firm leverage is increasing in firm size. Rajan and Zingales (1995) documented this positive relationship for publicly traded firms for several OECD countries, including the U.S. Recently, Dinlersoz, Kalemli-Ozcan, Hyatt, and Penciakova (2019) have shown that the positive relationship between leverage and firm size also extends to private U.S. firms. Extant models of firm leverage, however, do not explain this fact. In macroeconomics, the most well-known model of firm leverage, Cooley and Quadrini (2001), predicts a negative relationship between size and leverage.[1] In finance, canonical models of firm capital structure (Leland (1994), Leland and Toft (1996)) are solved under assumptions that imply that optimal leverage is constant and independent of firm size.[2]

The main goal of this paper is to explain the positive relationship between leverage and firm size when growth of existing firms and the entry of new ones is endogenous. To endogenize firm growth and firm entry, we assume a steady arrival of ideas for new product varieties. Each new idea is initially owned by someone and the owner can either sell the idea to an existing firm, leading to the growth in output of that firm, or use the idea in a startup, leading to the entry of a new firm.

To get a positive relationship between firm size and leverage, we assume that existing products can go extinct with a constant probability. This implies that a firm that manages more varieties has a less volatile growth rate. Firms can borrow with the option to default on their debts. Since output growth of larger firms is less volatile, their probability of default on any given level of debt is lower and, under certain conditions, this implies that larger firms will choose to be more leveraged.

A striking feature of our explanation of the leverage and firm-size relationship is its implications for how interest rates affect the startup rate and business concentration. We show, both analytically (in a stripped-down version of the model) and numerically, that in our model, a decrease in the risk-free rate increases the *financial synergy* from a firm's acquisition of a new variety, and does so *more* for larger firms because they are more leveraged. Thus, a decline in the risk-free rate results in more new varieties being bought by larger firms, which leads simultaneously to a decline in the startup rate and to a rise in business concentration (share of sales accounted for by larger

[1] In their model, growth stems from capital accumulation and finance is needed for investment. In this framework, diminishing returns to capital imply a negative – not positive – relationship between firm size and leverage.

[2] Of course, it is the case that optimal leverage depends negatively on the volatility of the firm's fundamental cash flow. More to the point, optimal capital structure models do not typically address the fact that businesses *choose* their size (Miao (2005) is an exception).

firms). Since the risk-free rate has in fact been declining for at least two decades, our model raises the possibility that the concomitant rise in business concentration and declines in startup rates — phenomena that have received a great deal of separate attention — may have a common cause in falling interest rates.

Our model is deliberately bare bones, but its key elements are grounded in well-established facts. The model relies on the volatility of output or employment growth being lower for larger firms, for which there is very good evidence. Stanley, Amaral, Buldyrev, Havlin, Leschhorn, Maass, Salinger, and Stanley (1996, Figure 2, p. 805) document that among publicly traded manufacturing firms that survive from one period to the next (so exit is ignored), the standard deviation of the growth rate of sales falls with firm size (see also Buldyrev, Pammolli, Riccaboni, and Stanley (2020, Figure 2.8, p. 21)). In section 5.3.2 we use the Census Bureau's Business Dynamics Statistics database to show that volatility of employment growth is declining in firm size. Importantly, our measure of employment growth volatility takes exit into account. Additionally, Davis, Haltiwanger, Jarmin, Krizan, Javier, Nucci, and Sandusky (2007, Figure 12, p. 41) document that a closely-related measure of revenue growth volatility (also based on the BDS) declines with firm size.[3]

The fact that larger businesses have less volatile growth rates does not automatically imply that they will have higher leverage. For this, it has to be easier for larger firms to borrow. We model this in two ways. In the main text, we assume that when firms borrow, they must obey a default probability constraint. The motivation for this constraint is the well-established fact that risky borrowers are simply denied credit.[4] In Appendix B, we explore a case where there is no constraint on default probability but default imposes a fixed cost on creditors. Since volatility of the growth rate of cash flow falls with firm size, either approach generates a positive association between leverage and firm size.

Our model embeds a theory of firm entry that recognizes that ideas for new products occur to *people* and they get to choose the organizational form in which to implement them (Chatterjee and Rossi-Hansberg (2012), Zábojník (2019)).[5] The theory applies when a person is contemplating

[3] Relatedly, Decker, D'Erasmo, and Boedo (2016) show that diversification — measured as the number of markets a firm is exposed to — is pro-cyclical and it is the larger firms that respond more in this way.

[4] The most direct evidence on this comes from surveys of small business lending: Federal Reserve Bank of New York (2017, p. 19) reports that among firms aged less than 5 years, 69 percent of those that requested credit received less credit than they sought; this percentage rises to 85 among businesses classified as medium/high risk. And even for investment banks, reputational concerns constrain the riskiness of the bonds they underwrite (Fang (2005)).

[5] In both of these earlier papers, the choice depended on the quality of the idea, with lower-quality ideas being sold and higher-quality ideas leading to startups. In contrast, in this paper the choice depends on the technological and financial synergies between the idea and existing firms.

setting up a pizza store and chooses between proceeding independently or as a franchise of a nationally-known brand. Similarly, the founders of spinoffs — new ventures that originate out of an existing one and are in the same line of business as the parent — often make a such a choice.[6] The other side of the coin, of course, are the many instances in which employees with new ideas implement their ideas in the firm in which they work. One indication that this occurs is the common practice of patent assignment, which transfers patent (or patent application) rights from inventor-employees to their employers. Another indication is the fact that, as assumed in this paper, individuals with valuable knowledge/ideas are compensated in the form of equity claims to future cash flows (Eisfeldt, Falato, and Xiaolan (2021)).

Our model is abstract in that firm growth is modeled as occurring via the addition of new product lines, rather than through productivity growth of existing product lines.[7] While it is clear that stability of sales growth improves with size, not much is known about why this is the case. But large firms are generally viewed as having more diversified sources of revenue and "growth by new product lines" is a simple way to model this view. In addition, this approach establishes a bridge to theories of endogenous growth (Romer (1990), Grossman and Helpman (1991), Aghion and Howitt (1992)) in which growth occurs via the arrival and absorption of new goods (varieties). While we don't model the production of new ideas, we consider a different aspect: Once the idea for a new good comes about, is it implemented in a startup or in an existing business and how is this choice affected by the risk-free rate? The choice has implications for the distribution of output across *firms* (business concentration), an important aspect of growth.[8]

Turning to the decline in the risk-free rate, the view that emerges from the many studies that have examined its possible causes (Caballero, Farhi, and Gourinchas (2008), Mendoza, Quadrini, and Ríos-Rull (2009), Eichengreen (2015), Del Negro, Giannone, Giannoni, and Tambalotti (2017), Farhi and Gourio (2018), among others) is that it has resulted largely from a rise in the premium placed on safety and liquidity. Consistent with this, our model treats the decline in the risk-free rate as occurring due to a change in the preferences of lenders. Since we assume that lenders are

[6]For example, Klepper (2007) argues that startups in the early auto industry resulted from disagreements among employees about the viability of new ideas, which suggests that the new ideas *could* have been implemented in the parent firm but the idea's owner chose not to do so.

[7]We also abstract from the connection between leverage and capital accumulation that has been the focus of many papers, including Cooley and Quadrini (2001), Albuquerque and Hopenhayn (2004), Arellano, Bai, and Zhang (2012), Jermann and Quadrini (2012), Khan and Thomas (2013), Arellano, Bai, and Kehoe (2016), Gomes, Jermann, and Schmid (2016) and Corbae and D'Erasmo (2021).

[8]See Luttmer (2010, Section 3.4, p. 559) and Chatterjee and Rossi-Hansberg (2012) for an in-depth discussion of this point.

risk-neutral, the decline is modeled simply as a decline in their degree of impatience.[9] Regarding consequences of low interest rates, Gopinath, Kalemli-Ozcan, Karabarbounis, and Villegas-Sanchez (2017) argue that, due to financial frictions, the decline in interest rates led to an increase in the misallocation of capital and lower productivity in South Europe; Caggese and Perez-Orive (2019) argue that low interest rates put firms with intangible capital at a disadvantage; Liu, Mian, and Sufi (2022) explore the role of low interest rates in generating greater business concentration through a "strategic competition effect," while Kroen, Liu, Mian, and Sufi (2021) present causal evidence that declines in interest rates benefit large firms, which increase leverage and also conduct more cash acquisitions — findings that complement the mechanisms emphasized in our paper. However, unlike ours, these studies do not connect low interest rates to lower entry rates.

The causes of the decline in the startup rate — often described as a "decline in business dynamism" — and the rise in business concentration are active areas of research. Regarding the decline in the startup rate, Hathaway and Litan (2014) list several factors, including slowing population growth, increasing business consolidation, and the rising burden of regulation and taxes as potential causes. The role of slowing of labor force growth has been stressed in Karahan, Pugsley, and Şahin (2019) and in Hopenhayn, Niera, and Singhania (2018), and that of changes in corporate tax rates in Neira and Singhania (2017). Studies that attempt to explain the rise in business concentration have examined increases in market power (De Loecker and Eeckhout (2020)) and the entry of large firms into new geographic markets (Hsieh and Rossi-Hansberg (2021), Rossi-Hansberg, Sarte, and Trachter (2020)).

In recent work, Aghion, Bergeaud, Boppart, Klenow, and Li (2019) and Akcigit and Ates (2019) explore, like us, a common cause for the decline in entry rates and the rise in concentration. The former focus on technological change that is increasingly benefiting larger firms and the latter on a decline in knowledge diffusion from leading to lagging firms. Although the conceptual frameworks of these studies are quite different from ours, they all share the key commonality that new ideas/products are increasingly appearing within larger firms.

The paper is organized as follows. Section 2 briefly documents the leverage-firm-size relationship that motivates the paper. Section 3 lays out our model of firm dynamics with borrowing, default,

[9] The assumption of risk neutrality is a significant simplification. Farhi and Gourio (2018) note that the decline in the risk-free rate cannot only be a result of a decline in the discount rate of investors as that would imply a very large counterfactual increase in price-to-dividend ratios. Most likely, the drop in the risk-free rate was also accompanied by a increase in the price of risk that kept equity valuations relatively stable.

entry and exit. In Section 4, a simple stripped-down version of this model is analyzed to explain the key idea of this paper: the market value of debt can affect the entry rate of new firms and the growth rate of existing firms. Section 5 analyzes the full model quantitatively and establishes that the key results derived in the stripped-down model carry over. This section also assesses the degree to which the quantitative model is in consonance with untargeted facts. Section 6 analyzes the impact of a decline in the risk-free rate on firm dynamics and shows that the model is capable of accounting for most of the decline in the entry rate since the late 1990s and also sheds some light on related trends over this period. Section 7 concludes. Finally, Appendixes A through D contain additional materials on facts and theory. In particular, Appendix C describes a variant of the model in which creditors incur a fixed default cost in the event of bankruptcy and shows that this alternative setup has properties similar to the main model.

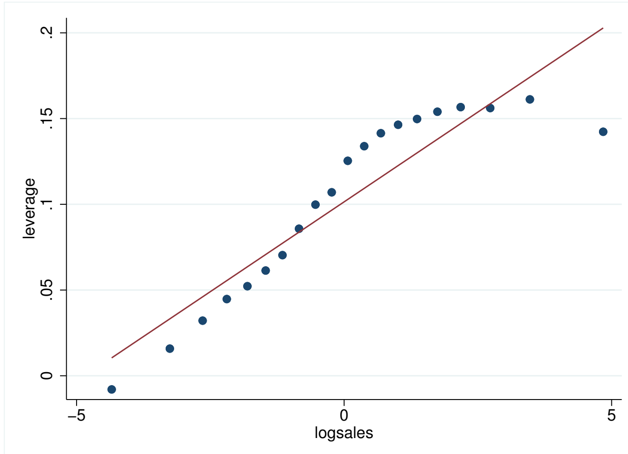
2 The Positive Relationship Between Leverage and Size

The positive relationship between firm size and firm leverage is a well-known regularity in finance. For the sake of background and completeness, we document this fact among publicly listed U.S. firms. As noted in the introduction, the positive relationship is true for private firms as well (but the data to demonstrate it is less easily available).

Figure 1 shows a (binned) scatter plot of firm leverage against firm size. Firm leverage is the ratio of a firm's total debt net of cash to its total assets; a firm's size is the deviation of the firm's log annual sales from the average (over all firms) in that year. There are upward of $175,000$ firm-year observations underlying the figure, so we plotted averages corresponding to 20 size bins. Median and average leverage is 0.094 and 0.076, respectively, and the slope of the line of "best fit" shown is 0.023.

The figure shows the "raw" pattern of association between leverage and size. In Appendix A, we present cross-sectional and panel regressions that include controls standard in the empirical literature and use four common measures of leverage. These measures differ in what is counted in debt and whether assets are in market or book values. Regardless of details, there is a highly statistically significant positive relationship between leverage and log sales; the magnitude of the response of leverage to size ranges from 0.013 to 0.038. As a point of comparison, our estimates are comparable to estimates reported in Dinlersoz, Kalemli-Ozcan, Hyatt, and Penciakova (2019,

Figure 1:
Leverage and Firm Size



Source: Authors' calculations using COMPUSTAT (1978-2015), Compustat data from S&P Global Market Intelligence, retrieved from Wharton Research Data Service (WRDS) on 12-15-21. Leverage is the ratio of debt net of cash to total assets recorded at market values. For each year, firms included are those (i) reporting in U.S. dollars, (ii) with book value, market value, and sales at least \$1 million in 2015 (the GDP deflator is used to determine the equivalent nominal cutoffs for earlier years), (iii) with nonnegative debt and cash holdings, (iv) with leverage between -1 and $+1$, and (v) in sectors other than Utilities and Financials.

Table 4). They estimate a response of leverage to log sales of 0.0178 for publicly-listed firms and a response of 0.0281 for private firms.

We turn now to a model of firm borrowing with option to default that can generate this positive association.

3 Model

3.1 The Environment

The economy is composed of a fixed continuum of workers and a potentially changing continuum of heterogeneous firms. Time is discrete and a firm is defined by the pair (K, B) where K is the number of varieties owned by the firm and B is its debt, both referring to beginning-of-period quantities.

There is an upper bound $K_{\max}$ to the number of varieties that a single firm can manage, stemming from "span of control" limitations (Lucas, Jr. (1978)), so $K \in \mathbb{K} \equiv \{1, 2, 3, \ldots, K_{\max}\}$. There is also an upper bound $B_{\max}$ on the amount of debt a firm can carry but if $B_{\max}$ is sufficiently large, this bound is nonbinding. In the theory, firms will not have an incentive to save and so it is assumed that $B \in [0, B_{\max}]$ without loss. To minimize technicalities, $[0, B_{\max}]$ is approximated by a finite set (of grid points) $\mathbb{B}$.

Each period is divided into two subperiods. In the first subperiod, ideas arrive and are implemented; in the second subperiod, a fraction of existing products become extinct and the surviving varieties are produced. Each variety is produced with a linear technology using only labor. Firms compete monopolistically to earn a constant markup over marginal (labor) cost for each unit of each variety produced. Following production, firms partially or fully repay their debt and (potentially) issue new debt. Consumption occurs at the end of the period.

At the start of a period, the aggregate state is composed of two components. The endogenous component is the vector $\{\mu(K, B), (K, B) \in \mathbb{K} \times \mathbb{B}\}$, where $\mu(K, B) \geq 0$ is the mass of firms of type (K, B). The exogenous component is the sequence $\mathbf{r} = \{r_{t+1}, r_{t+2}, r_{t+3} \ldots\}$ of current and future real interest rates which decision makers perfectly anticipate and take as given. The pair $\{\mu, \mathbf{r}\}$ is denoted X. Then, $\mu' = \Gamma_\mu(X)$ is the perceived law of motion of μ and $\mathbf{r}' = \Gamma_r(X)$ is exogenous law of motion of $\mathbf{r}$.[10] Collectively, the law of motion of X is denoted $\Gamma(X)$.

3.2 Arrival of New Ideas and Their Implementation

It is assumed that ideas occur to workers and that the measure of new ideas arriving each period is constant and given by $M > 0$. Thus, we do not model the *process* of idea generation, which has been the focus of a vast empirical and theoretical literature, and instead focus whether a new idea is implemented in a startup or an existing firm.

An important feature of the model is the way knowledge of newly arriving ideas is distributed across existing firms. A firm that owns K varieties encounters ideas for new varieties at the rate ρK. The proportional relationship between the number of ideas encountered and K is in the spirit of the firm-size distribution literature wherein the expected rate of growth of varieties (or of "blueprints," in the terminology of Luttmer (2010)) is generally independent of the number of varieties ("blueprints") owned by the firm.[11]

[10] Γ_r is just the left shift operator that takes $\mathbf{r} = \{r_{t+1}, r_{t+2}, r_{t+2} \ldots\}$ to $\mathbf{r}' = \{r_{t+2}, r_{t+3}, r_{t+4} \ldots\}$.

[11] If every new variety encountered by a firm of size K is successfully absorbed, the expected (gross) growth rate of the number of varieties in the firm (prior to product survival shocks) is $[\rho K(K+1) - (1 - \rho K)K]/K = (1 + \rho)$.

The rate per variety, ρ, is taken as given by all decision makers but its value depends on the aggregate state X. The reason is that by definition $\rho = M/N$, where N is the measure of varieties in existence at the start of the period and is given by $\sum_K K \sum_B \mu(K, B)$,[12] which depends on μ and, hence, on X. In what follows, we recognize this dependence as $\rho(X)$.

To keep the exposition streamlined, in the main body of the paper we assume that the period is short enough, and so $\rho(X)$ small enough, so that $\rho(X)K_{\max} < 1$. Then $\rho(X)K$ can be interpreted as a probability and a firm can get matched with at most one idea per period. In the numerical analysis, $\rho(X)K_{\max}$ is allowed to exceed 1.[13]

To incorporate a choice between selling an idea to an existing firm and implementing it in a startup we assume that an idea is potentially valuable to some firm in the economy. In this (randomly chosen) firm, the idea can be successfully turned into a new variety with probability $s \in [0, 1]$. For each idea-firm match, the value of s is drawn independently from a distribution $F(s)$. A higher s indicates greater *technological synergy* between the idea and the firm's existing capabilities. We also assume that an idea can always be implemented in a startup and, if it is, it's successfully transformed into a new variety with a constant probability $\sigma \in (0, 1)$.[14]

Whether the idea is implemented in an existing firm or in a startup is determined via bargaining between the firm and the worker who got the idea.[15] Let $W(\widetilde{K}, B; X)$ denote the value of a firm that at the end of the first subperiod owns $\widetilde{K}$ varieties and has debt B. For a firm that starts the period with $K < K_{\max}$ there is a potential gain from implementing a new idea with success probability s if

$$s[W(K + 1, B; X) - W(K, B; X)] \geq \sigma W(1, 0; X). \tag{1}$$

The l.h.s. is the expected gain to the firm if it implements the idea for a new variety. The r.h.s. is the expected gain to a startup which by definition owns one variety and has no legacy debt. If

[12]For a given K, the inner sum over B gives the measure of varieties owned by all firms owning K varieties and the outer sum over K gives the total measure of varieties.

[13]When $\rho K_{\max} > 1$, a firm that owns K varieties encounters $\lfloor \rho(X)K \rfloor$ ideas for sure and the $\lfloor \rho(X)K \rfloor + 1^{th}$ idea with probability $\rho(X)K - \lfloor \rho(X)K \rfloor$ (here $\lfloor \rho(X)K \rfloor$ is the largest integer less than or equal to $\rho(X)K$). The decision problem corresponding to this general case is stated in Appendix B.

[14]The setup is similar to a model with search frictions in that we permit the worker to negotiate with only one firm.

[15]In the general case where a firm receives multiple ideas, we assume that the bargaining proceeds sequentially: The firm bargains with a randomly chosen inventor who is matched with the firm, knowing only the probability distribution of how many more inventors are in line to bargain and knowing whether previously purchased ideas have been successfully turned into new varieties.

the surplus is nonnegative, we assume that the new variety is implemented by the firm and the inventor gets $\sigma W(0,1;X)$ through shares in the existing firm.[16,17]

The core idea of the paper is that the difference $W(K+1,B;X) - W(K,B;X)$ is increasing in K. This feature stems from the fact that a larger firm has a lower volatility of sales growth and is, therefore, less risky and able to borrow a larger fraction of its future cash flow (and would like to do so). This makes a larger firm more willing to purchase an idea and this is more true when the risk-free rate is lower. Section 4 exposits this core idea in the context of a simplified version of the model.

Let $s^*(K,B;X)$ solve (1) with equality, so $s^*(K,B;X)$ can exceed 1 or fall below 0. If $K < K_{\max}$ and $s \geq s^*(K,B;X)$ the idea is implemented in the firm; if $s < s^*(K,B;X)$ it is implemented in a startup. If $K = K_{\max}$, the idea is implemented in a startup regardless of the value of s.

Under these assumptions, the start-of-the-period value of the firm, denoted $Z(K,B;X)$, is

$$Z(K,B;X) = \tag{2}$$

$$\begin{cases} W(K,B;X) & \text{if } K = K_{\max} \\ \rho(X)K \int \mathbb{1}_{\{s \geq s^*(K,B;X)\}} \{s[W(K+1,B;X) - W(K,B;X)] \\ \qquad -\sigma W(1,0)]\} dF(s) + W(K,B;X) & \text{if } K < K_{\max} \end{cases} \tag{3}$$

If the firm is already at $K_{\max}$ then Z is the same as W. Otherwise, Z takes into account the expected gain from encountering an idea with a high enough s. Note that the probability that the firm acquires a new variety is $\rho(X) \int s \mathbb{1}_{\{s \geq s^*(K,B;X)\}} dF(s)$ and, so, 1 minus this probability is the probability of the firm not aquiring a new variety.

3.3 Destruction of Varieties

Let $\widetilde{K}$ denote the number of varieties owned by the firm, including any that it bought in the current period and successfully integrated into its portfolio. At this juncture, a variety can become extinct

[16] If the firm has unit shares outstanding, it issues a additional units to the inventor, where $a = \sigma W(1,0;X)$. The value of $1+a$ units of shares is $s[W(K+1,B;X) - W(K,B;X)] + W(K,B;X)$ and the post-purchase value of the original unit share is $s[W(K+1,B;X) - W(K,B;X)] + W(K,B;X) - \sigma W(1,0;X)$.

[17] If, instead, we assumed that inventors must be compensated in cash, firms would have an incentive to accumulate cash balances to purchase ideas (fund investment opportunities), which we do see firms doing. We note, however, that it is possible for a 1-variety firm to not choose to accumulate any cash (if the interest rate is too low or investment opportunities arrive too rarely). In that case, the long-run equilibrium would feature only 1-variety firms and firms would not grow over time.

with probability ϕ. The probability that a firm with $\widetilde{K}$ varieties ends up with $0 \leq K' \leq \widetilde{K}$ varieties is, therefore,

$$h(\widetilde{K}, K') = \binom{\widetilde{K}}{K'} (1-\phi)^{K'} \phi^{\widetilde{K}-K'}.$$

3.4 Production and Profits

Following the extinction shocks, the firm undertakes production if it has any surviving varieties. Let N' denote the measure of varieties in existence in the economy at the time of production. A consumer's preferences over consumption of varieties is given by the CES aggregator

$$U = \left[\int_0^{N'} c(n)^\gamma dn \right]^{1/\gamma}, \quad \gamma \in (0,1). \tag{4}$$

Let $p(n)$ denote the nominal (dollar) price of variety n and let I denote nominal aggregate expenditure on all varieties. Then, the demand function for variety n is

$$\left[\frac{p(n)}{P} \right]^{-\frac{\gamma}{1-\gamma}} \left[\frac{I}{P} \right], \quad \text{where } P = \left[\int_0^{N'} p(n)^{-\frac{\gamma}{1-\gamma}} dn \right]^{-\frac{1-\gamma}{\gamma}}. \tag{5}$$

Firms can hire workers at the nominal wage W. If the productivity of labor is $\alpha > 0$, the nominal marginal cost of producing a variety is W/α. Profit maximization implies that $p^* = (1/\gamma)W/\alpha$ and flow nominal profits from producing the optimal output q^* is $\Pi = [1-\gamma]p^* q^*$. Since all varieties appear symmetrically in U, the profit-maximizing price is the same for all varieties. Then, the flow of real profits is

$$\pi(N'|q^*) = \frac{\Pi}{P} = [1-\gamma] N'^{\frac{1-\gamma}{\gamma}} q^*. \tag{6}$$

In general equilibrium, the output of each variety must be such as to absorb, in the aggregate, all available labor. Letting L denote the constant measure of workers, we must have

$$q^* N' = \alpha L. \tag{7}$$

Hence, we obtain that in general equilibrium the optimal real profit flow is

$$\pi(N') = [1-\gamma] \alpha L N'^{\frac{1}{\gamma}-2}. \tag{8}$$

The measure of varieties in existence following the extinction shocks is also the measure of varieties at the start of the next period. Thus, $N' = \sum_K K \sum_B \mu'(B,K)$. Since $\mu' = \Gamma_\mu(X)$, the optimal equilibrium real profit is a function of X via the perceived law of motion of X:

$$\pi(X) = [1-\gamma]\alpha L \left[\sum_K K \sum_B \Gamma_\mu(X)(B,K)\right]^{\frac{1}{\gamma}-2}. \tag{9}$$

3.5 Debt, Default, and Exit

Following production, the firm services any existing debt if it can and potentially engages in new debt issuances. We assume that a firm's debt is an obligation in real terms and, so, B and $\pi(X)$ are in the same units. If the firm issues new debt B', the real price at which it sells this debt is $q(K', B'; X)$.[18] Both K and B are relevant for assessing the probability of default on debt and, so, both appear as arguments of the bond price. The price also depends on the aggregate state X directly through the sequence $\mathbf{r}$ and indirectly through the implied sequence of future μ's which affect future profit flows and default probabilities.

We impose two constraints on a firm's debt decisions. First, B' must respect nonnegative dividend payouts (equivalent to nonnegative consumption)

$$\pi(X)K' - B + q(K', B'; X)B' \geq 0. \tag{10}$$

If the firm's cash $\pi(X)K'$ falls short of its obligations B, its only option is to meet the deficit by debt issuance. Effectively, the constraint imposes a lower bound on B'. And, second, B' must respect a default probability constraint:

$$d(K', B'; X') \leq \theta,\ \ \theta \in (0,1] \text{ and } X' = \Gamma(X). \tag{11}$$

Here $d(K', B'; X')$ is the probability of default next period on debt issued in the current period. As mentioned in the introduction, the motivation for this constraint is the well-established fact that lenders deny credit to risky firms. In Appendix B we consider the alternative case where default imposes a fixed cost on lenders.

[18] The obligation to pay 1 unit next period is the obligation to pay P' in nominal terms; the current nominal price of this obligation is $q(\cdot)P$.

A firm is in default if it cannot meet its debt obligations fully. To formalize this, let $G(K'; X)$ be the maximum revenue from bond sales consistent with the bonds meeting the default probability constraint (11). That is,

$$G(K'; X) = \max_{B' \in \mathbb{B}} q(K', B'; X)B' \tag{12}$$

s.t.

$$d(K', B'; X') \leq \theta \text{ and } X' = \Gamma(X).$$

Define

$$\bar{B}(K'; X) = \pi(X)K' + G(K'; X). \tag{13}$$

Then $\bar{B}(K'; X)$ is the maximum amount of resources available to the firm and default occurs if and only if $B > \bar{B}(K'; X)$. Thus, the default decision rule $D(K', B; X)$ is given by

$$D(K', B; X) = \begin{cases} 1 & \text{if } B > \bar{B}(K'; X) \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

In the event of default, creditors receive $\bar{B}(K'; X)$. The value of a firm in default is then given by

$$V^D(K', B; X) = \max_{B' \in \mathbb{B}} \pi(X)K' - \bar{B}(K'; X) + q(K', B'; X)B' + \beta Z(K', B'; X') \tag{15}$$

s.t.

$$q(K', B'; X)B' = G(K'; X), \; d(K', B'; X') \leq \theta \text{ and } X' = \Gamma(X).$$

Here $0 < \beta < 1$ is the discount factor of owners and workers. If there is a unique B' that attains $G(K'; X)$, then that is the only choice available to the firm. In any event, the dividend payout in default is always zero but owners retain rights over the firm's future cash flow. Thus, default resembles a Chapter 11 business reorganization rather than a business liquidation.

If the firm does not default, that is $B \leq \bar{B}(K'; X)$, the firm solves

$$V^R(K', B; X) = \max_{B' \in \mathbb{B}} \pi(X)K' - B + q(K', B'; X)B' + \beta Z(K', B'; X') \tag{16}$$

s.t.

$$\pi(X)K' - B + q(K', B'; X)B' \geq 0, \;\; d(K', B', X') \leq \theta \text{ and } X' = \Gamma(X).$$

We denote the firm's optimal choice of debt by $B(K', B; X)$.

Exit of a firm occurs if it loses all its varieties. This is because if $K' = 0$, the firm cannot acquire new varieties (since $\rho(X) \times 0 = 0$) and 0 becomes an absorbing state. Since $K'' = 0$ with certainty, the probability of default on any amount of debt is 1 and, so, $\bar{B}(0; X) = 0$. Hence, if $B > 0$, the exiting firm defaults and creditors get nothing.

We can now give the expression for the probability of default on bonds issued in the current period conditional on X':

$$\begin{aligned} d(K', B'; X') = [\rho(X')K' \int s \mathbb{1}_{\{s \geq s^*(K', B'; X')\}} dF(s)] \mathbb{E}_{(K''|K'+1)} D(K'', B'; X') + \\ [1 - \rho(X')K' \int s \mathbb{1}_{\{s \geq s^*(K', B'; X')\}} dF(s)] \mathbb{E}_{(K''|K')} D(K'', B'; X'). \end{aligned} \tag{17}$$

The first term is the product of the probability that the firm successfully adds a new variety next period with the probability of default conditional on having done so.[19] The second term is the complementary case where the firm fails to add a new variety. Since $X' = \Gamma(X)$, this default probability is known to all decision makers in the current period.

Finally, we give the expression for $W(K, B; X)$:

$$W(K, B) = \mathbb{E}_{(K'|K' \leq K)} \left[[1 - D(K', B; X)] V^R(K', B; X) + D(K', B; X) V^D(K', B; X) \right]. \tag{18}$$

[19]The full expression for the probability of default conditional on acquiring new variety next period is

$$\sum_{K''=0}^{K'+1} h(K'+1, K'') \mathbb{1}_{\{(B' > \bar{B}(K''; X'))\}},$$

which is equivalent to $\sum_{K''=0}^{K'+1} h(K'+1, K'') D(K'', B'; X')$ (similarly for the probability of default conditional on not acquiring a variety).

The model analog of leverage of a firm in state (K, B) is

$$\frac{B}{B + Z(K, B)}. \tag{19}$$

It is the ratio of the value of obligations due this period to the total value of a firm as represented in the data, namely, the firm's debt obligations plus its beginning-of-period value to owners.

3.6 Evolution of Firm Distribution

In this (sub) section, we describe the equations that govern the evolution of the distribution of firms over the state space, namely $\Gamma_\mu(X)$. For this, it is helpful to introduce notation for the distribution of firms at the end of the first subperiod, i.e., after all decisions and outcomes regarding the implementation of new varieties have occurred but the extinction shocks have not. Denote this interim distribution by $\tilde{\mu}(K, B)$.

Then, $\Gamma_\mu(X)$ has the following simple form

$$\Gamma_\mu(X)(K', B') = \sum_{B \in \mathbb{B}} \sum_{K' \leq \tilde{K}} \left[h(\tilde{K}, K') \mathbb{1}_{\{B' = B(K', B; X)\}} \right] \cdot \tilde{\mu}(\tilde{K}, B; X), \; (K', B') \in \mathbb{K} \times \mathbb{B}. \tag{20}$$

On the r.h.s. the term in $[\cdot]$ is the product of the probability that a firm that starts the second subperiod with $\tilde{K}$ varieties ends up with $K' \leq \tilde{K}$ varieties and the probability (which is either 0 or 1) that a firm in state (K', B) chooses B'. The product of this joint probability with the measure of firms on the grid $(\tilde{K}, B')$ gives the measure of firms arriving on grid (K', B') from this particular source; summing over all relevant sources yields the total measure of firms on the grid (K', B') at the start of the next period. Note that since since $0 \notin \mathbb{K}$, this equation records the flows of surviving firms only.

To complete the description of $\Gamma_\mu(X)$, we now describe how $\tilde{\mu}$ evolves from μ. To begin,

$$\tilde{\mu}(1, 0; X) = \sigma \sum_{B \in \mathbb{B}} \sum_{K \in \mathbb{K}} \left[\rho(X) K \int \mathbb{1}_{\{s < s^*(K, B; X)\}} dF(s) \right] \cdot \mu(K, B) + \left[1 - \rho(X) \int s \mathbb{1}_{\{s \geq s^*(1, 0; X)\}} dF(s) \right] \cdot \mu(1, 0). \tag{21}$$

The l.h.s is the measure of firms with one variety and no debt at the end of the first subperiod. On the r.h.s., the first term (with the double sum) is the measure of new firms created.[20] The second term is the measure of existing firms that started the period with 1 variety and 0 debt and failed to acquire a new variety.

Next,

$$\tilde{\mu}(1, B; X) = \left[1 - \rho(X) \int s \mathbb{1}_{\{s \geq s^*(1,B;X)\}} dF(s)\right] \cdot \mu(1, B), \quad B \in \mathbb{B} \setminus \{0\} \tag{22}$$

gives the measure of firms that have 1 variety and some $B > 0$; it consists of all firms that started the period with 1 variety and that B and failed to add a new variety.

Finally,

$$\begin{aligned}\tilde{\mu}(K, B; X) = &\left[\rho(X)(K-1) \int s \mathbb{1}_{\{s \geq s^*(K-1,B;X)\}} dF(s)\right] \cdot \mu(K-1, B) + \\ &\left[1 - \rho(X)(K) \int s \mathbb{1}_{\{s \geq s^*(K,B;X)\}} dF(s)\right] \cdot \mu(K, B), \quad B \in \mathbb{B},\ K \in \mathbb{K} \setminus \{1\}\end{aligned} \tag{23}$$

gives the flow into states with $K \geq 2$ and $B \geq 0$. It consists of firms that started with $K-1$ varieties and added a variety and those that started with K varieties and did not.

3.7 Equilibrium

An equilibrium is an initial X_0, a function $\rho^*(X)$, decision rules $s^*(K, B; X)$, $D^*(K', B; X)$, and $B^*(K', B; X)$, value functions, $V^{*D}(K', B; X)$, $V^{*R}(K', B; X)$, and $G^*(K'; X)$, a price function $q^*(K', B'; X)$, and a perceived law of motion $\Gamma^*(X)$ such that decision rules and value functions solve the relevant optimization problems given ρ^*, q^*, Γ^* and the following three conditions hold:

1. *Consistency of* ρ:

$$\rho^*(X) = \frac{M}{\sum_K K \sum_B \mu(B, K)}. \tag{24}$$

[20]The term in square brackets is the joint probability that the firm in state (K, B) encounters a new variety with an acceptable s; multiplication by $\mu(K, B)$ gives the measure of startup attempts from such firms and the sum over (K, B) yields the total measure of startup attempts. Multiplication by σ then gives the measure of startup attempts that successfully end in a new firm.

2. Bond prices yield the risk-free rate in expectation:

$$\begin{aligned} q^*(K', B'; X)(1 + r'(X)) = & \left[\rho(X')^* K' \int s \mathbb{1}_{\{s \geq s^*(K', B'; X')\}} dF(s)\right] \times \\ & \mathbb{E}_{(K''|K'+1)} \left[[1 - D^*(K'', B'; X')] + D^*(K'', B'; X') \frac{\bar{B}(K''; X', G^*)}{B'}\right] + \\ & \left[1 - \rho(X')^* K' \int s \mathbb{1}_{\{s \geq s^*(K', B'; X')\}} dF(s)\right] \times \\ & \mathbb{E}_{(K''|K')} \left[[1 - D^*(K'', B'; X')] + D^*(K'', B'; X') \frac{\bar{B}(K''; X', G^*)}{B'}\right], \end{aligned} \quad (25)$$

where $r'(X)$ is the risk-free rate between the current period and the next period (in calendar time, r_{t+1}) and $X' = \Gamma(X)$. The l.h.s. is the opportunity cost of investing in a unit bond with price $q(K', B'; X)$, and the r.h.s. is the expected payoff from doing so. The first term on the r.h.s. is the product of the probability of acquiring a new variety and the expected payoff on the bond conditional on having added a new variety. The payoff is 1 if there is no default, and it is the repayment ratio $\bar{B}(K''; X')/B'$ if there is default. The second term is similar, covering the case where a new variety is not acquired. Since there is generally a payment in the event of default, $q(K', B'; X) \geq [1 - d(K', B'; \Gamma(X))]/(1 + r')$.

3. Perceived law of motion coincides with the actual law of motion given by equations (20) - (23).

4 External Finance and Value of New Ideas in a Two-Period Model

The goal of this section is to explain, in the context of the stripped-down version of the model, how access to external finance makes new ideas more valuable to larger firms.

Consider a version in which $K_{\max} = 2$. Thus, a firm can own one or two varieties. In addition, assume that there are only two periods: the current period and a final period. The timeline of events in the current period is as described in section 3.1, except that profit from each variety is exogenously given and equal to π. The final period is a "dummy" period in which only extinction shocks occur, followed by realization of profits, debt payments and dividends.

Following the notation conventions of the main model, we use K and B to denote the varieties and debt inherited from the past; $s^*(K, B)$ to denote the threshold value of s below which a firm rejects an idea for purchase; B' to denote the debt incurred in the current period; K' and K'' to denote the number of varieties owned at the end of the current and final periods, respectively; and $q(K', B')$ to denote the price of debt issued in the current period.

In this two-period world, default on debt incurred in the current period occurs if and only if $\pi K'' < B'$. Thus, the default probability on debt B' issued by a K'-variety firm is $d(K', B') = \Pr\{\pi K'' < B'|K'\}$.

The most that a K'-variety firm can possibly borrow depends on θ. If $\theta < 1$, a firm cannot borrow more than the maximum possible cash flow next period because if it did, the probability of default would be 1 and that would exceed θ. Since $K'' \leq K'$, $B' \leq \pi K'$ for $\theta < 1$. For $\theta = 1$, $B' > \pi K'$ is possible, but committing more than what the firm can ever pay simply leads to a reduction in prices with no change in revenue from bond sales. So, without any loss of generality, we may assume that $B' \leq \pi K'$ for all $0 \leq \theta \leq 1$.

The next three propositions tell us what happens for different ranges of θ values. Proposition 1 deals with middle range of θ for which leverage is increasing in size, which is the most relevant case with respect to the data and the quantitative results of the next section. Propositions 2 and 3 deal with the higher and lower ranges of θ, respectively.

Proposition 1. *Suppose* $(1 + r)\beta < 1$ *and* $\phi^2 \leq \theta < \phi$. *Then (i)* $B(K') = (K' - 1)\pi$*, (ii)* $s^*(1, 0) < \sigma$ *and is increasing in* r*, (iii)* $s^*(1, B)$ *is increasing in* B*.*

Proof. (i). Consider a firm with $K' = 1$ at the start of the second subperiod of the current period. If it borrows some amount $0 < B' \leq \pi$, it will default with probability ϕ. Since $\theta < \phi$, the default probability constraint is violated for any such B and the firm cannot borrow at all. So, $B(1) = 0$.

Consider a firm with $K' = 2$. If the firm borrows $\pi < B' \leq 2\pi$, it will default if it loses one or both varieties in the final period. Thus $d(2, B') = 1 - (1 - \phi)^2$. For these levels of borrowing to be feasible, it must be that $1 - (1 - \phi)^2 \leq \theta$, or, that $(1 - \theta) \leq (1 - \phi)^2$. But since $(1 - \theta) \geq (1 - \phi)$, this is impossible. So, borrowing in this range is not feasible.

If the firm borrows $0 < B' \leq \pi$, it will default if it loses both varieties, so $d(2, B') = \phi^2$. Since $\phi^2 < \theta$, these borrowing levels are feasible. Furthermore, since the firm defaults only when its cash flow is zero, it pays nothing to creditors in default. So, the bond price schedule it faces is

$$q(2, B) = \frac{1 - \phi^2}{1 + r} \quad 0 < B' \leq \pi. \tag{26}$$

What is the firm's optimal B' in this range? If the firm borrows B', it gets $q(2, B')B'$ in the current period and promises to repay B' next period with probability $[1 - \phi^2]$. Its net utility gain from issuing B' is $[1-\phi^2]B'[(1+r)^{-1}-\beta]$. Since $\beta(1+r)<1$, this net gain is strictly positive and the optimal debt level is the maximum feasible debt level π.

Thus, $B(K')=(K'-1)\pi$ for $K'\in\{1,2\}$.

(ii) Consider a firm that starts the current period with $(K,B)=(1,0)$. Since a variety survives with probability $(1-\phi)$ and a 1-variety firm cannot borrow, conditional on survival a 1-variety firm earns π in the current period and $(1-\phi)\pi$ in expectation in the final period. Thus, its value prior to the realization of the current period extinction shock is

$$W(1,0)=(1-\phi)\left[\pi+\beta\left(1-\phi\right)\pi\right]. \tag{27}$$

Next, consider a firm that starts the current period with $(K,B)=(2,0)$. Such a firm will borrow π if both its varieties survive in the current period, otherwise it does not (cannot) borrow. If both varieties survive in the current period and it borrows π, it will distribute dividends equal to π in the final period if *both* varieties survive in the final period; otherwise, its debt will equal or exceed its cash flow and dividends will be zero. So, its value prior to the realization of current period extinction shocks is

$$W(2,0)=(1-\phi)^2\left\{2\pi+\pi\left[\frac{1-\phi^2}{1+r}\right]+\beta(1-\phi)^2\pi\right\}+2(1-\phi)\phi\left\{\pi+\beta(1-\phi)\pi\right\}, \tag{28}$$

where we have used the fact that $q(2,\pi)=(1-\phi^2)/(1+r)$.

We may verify that the gain to the $(K,B)=(1,0)$ firm from acquiring a variety is

$$W(2,0)-W(1,0)=W(1,0)+\left[(1-\phi)^2\left(1-\phi^2\right)\pi\left(\frac{1}{1+r}-\beta\right)\right]. \tag{29}$$

The gain is composed of two parts: The first part is the stand-alone value of a firm with 1 variety and no debt, which is $W(1,0)$. In addition, if both varieties survive the extinction shocks in the current period, which occurs with probability $(1-\phi)^2$, the firm gets to issue π units of debt and the term multiplying $(1-\phi)^2$ is the net utility gain from issuing $B'=\pi$. Thus, the term in $[\cdot]$ is the financial benefit from combining two varieties in the same firm.

Since $s^*(1,0)$ is the s value for which a firm with $(K,B)=(1,0)$ is indifferent between acquiring a new idea or not, $s^*(1,0)$ satisfies

$$s^*(1,0)[W(2,0)-W(1,0)] = \sigma W(1,0). \tag{30}$$

Since $[W(2,0)-W(1,0)] > W(1,0)$, it follows that $s^*(1,0) < \sigma$. And, since $W(1,0)$ is independent of r it follows from (29) that $s^*(1,0)$ is positively related to r.

(iii). Consider a firm with $(K,B)=(1,B)$, $B \leq \pi$. For this firm, the gain from acquiring a new variety is

$$W(2,B)-W(1,0) = \left[(1-\phi)(\pi-\phi B)+\beta(1-\phi)^2\pi\right] + (1-\phi)^2\left(1-\phi^2\right)\pi\left(\frac{1}{(1+r)}-\beta\right). \tag{31}$$

$W(2,B)-W(1,0)$ differs from $W(2,0)-W(1,0)$ in that the term $(1-\phi)(\pi-\phi B)+\beta(1-\phi)^2\pi$ appears in place of $W(1,0)$ and is less than it by the quantity $-\phi(1-\phi)B$. This can explained as follows: In the event the firm's own variety fails but the one it acquired survives, which happens with probability $\phi(1-\phi)$, the firm must repay its debt. In other words, a portion of the expected increase in cash flow from an acquisition goes to the firm's *existing* creditors. Since $s^*(1,B)$ satisfies $s^*(1,B)[W(2,B)-W(1,0)] = \sigma W(1,0)$, it follows from (31) that $s^*(1,B)$ is increasing in B. □

Proposition 2. *Suppose* $(1+r)\beta < 1$ *and* $\phi \leq \theta \leq 1$*. Then (i)* $B(K') = \pi K'$ *for* $K' \in \{1,2\}$ *and (ii)* $s^*(K',0) = \sigma$ *for all* r*.*

Proof. (i) If $\phi \leq \theta$, the 1-variety firm can borrow positive amounts and a 2-variety firm can now borrow more than π without violating the default probability constraint. The bond price schedule for a 1-variety firm is

$$q(1,B') = \frac{(1-\phi)}{1+r} \quad 0 < B' \leq \pi \tag{32}$$

and that for a 2-variety firm is

$$q(2,B') = \begin{cases} \frac{1-\phi^2}{1+r} & \text{if } 0 < B' \leq \pi \\ \frac{(1-\phi)^2}{1+r} + \frac{2\phi(1-\phi)}{1+r}\frac{\pi}{B'} & \text{if } \pi < B' \leq 2\pi \end{cases} \tag{33}$$

Note that when the firm borrows more π, there is a payment to creditors in some default states (shared pro rata among creditors) that accounts for the second term in the bottom branch. Since $\beta(1+r) < 1$, one may easily verify that a 1-variety firm will borrow the maximum feasible amount π and 2-variety firm will similarly borrow 2π.

Hence $B(K') = \pi K'$ for $K' \in \{1, 2\}$.

(ii) Next, if $B(K') = \pi K'$, we may verify that $W(2, 0) = 2W(1, 0)$ regardless of the value of r. Hence $W(2, 0) - W(1, 0) = W(1, 0)$ and $s^*(1, 0) = \sigma$. □

Proposition 3. *Suppose* $0 \leq \theta < \phi^2$*. Then,* $B(K') = 0$ *for* $K' \in \{1, 2\}$ *and* $s^*(1, 0) = \sigma$*.*

Proof. For these levels of θ, neither a 1- nor a 2-variety firms can borrow since default probability on any level of borrowing will exceed θ. Hence $B(K') = 0$ for $K' \in \{1, 2\}$. Then, we may verify that once again $W(2, 0) = 2W(1, 0)$. Therefore, $W(2, 0) - W(1, 0) = W(1, 0)$ and $s^*(1, 0) = \sigma$. □

Proposition 1 shows that access to external finance makes a new idea more valuable to a firm, if a larger firm is able to borrow a bigger fraction of its current cash flow, i.e., if the ratio $B(K')/\pi K'$ is increasing in K'. The ratio $B(K')/\pi K'$ is closely related to firm leverage as $\pi K'$ is closely related to the total value of a firm with K' varieties. To see this, recall that the total value of a firm with K' varieties is the sum of its outstanding debt and the value of the residual cash flow to the owners. In the present context, the latter is just the expected value of cash flow not committed to creditors. Thus, the total value of the firm is

$$B(K') + \mathbb{E}_{(K''|K')} \max[\pi K'' - B(K'), 0]. \tag{34}$$

For a $K' = 1$ firm, this expression simplifies to $(1 - \phi)\pi$ as such a firm cannot issue any debt and its expected final period cash flow is just $(1 - \phi)\pi$; for a $K' = 2$ firm, it simplifies to $(1 - \phi)2\pi$. Thus, the total value of a firm scales linearly with $\pi K'$.

In Proposition 2, the default probability constraint is lax enough to permit both 1- and 2-variety firms to commit all future cash flow to creditors. While the larger firm can borrow more, equilibrium *leverage* is the same for both firms and equal to 1. In Proposition 3, neither a 1-variety nor a 2-variety firm can borrow, so equilibrium leverage is 0. In both cases, equilibrium leverage is independent of size and there is no financial benefit to a 1-variety firm from acquiring a new variety.

Thus, it is optimal for the 1-variety firm to purchase an idea if and only if there is a technological synergy, i.e., the success probability of the idea is higher in the firm than in a startup.

When there is a financial benefit from acquiring a new variety, the firm will purchase an idea even if its success probability is *lower* than the success probability of that idea in a startup, i.e., $s^*(1,0) < \sigma$ (Proposition 1 (ii)). From a social point of view, this is a *misallocation of resources*: Expected cash flow is reduced if ideas are implemented in the organization in which its success probability is lower. Furthermore, the misallocation is greater the lower is r as $s^*(1,0)$ declines with r. And, the lower is r the lower is the startup rate as fewer ideas are implemented in startups.

Finally, part (iii) of Proposition 1 shows that a firm with legacy debt is more choosy about the ideas it purchases. This is just a manifestation of the well-known *debt overhang effect*. In the presence of legacy debt, a portion of the financial benefit from an acquisition goes to existing creditors, which reduces the firm's incentive to spend resources to acquire a new variety.

4.1 Outside Equity

In the main model, we assumed that the only source of external finance is debt. In this two-period model we can investigate how matters change if firms can pay a fixed cost to access the equity market.

An *equity contract* is a promise by business owners to deliver future cash flows not committed to bond holders.[21] Let the fixed cost of accessing the equity market be $\kappa > 0$. As before, outside investors care only about the expected returns, and the opportunity cost of their funds is r. Then, the market value of an equity contract of a firm in state (K', B') is the second term in (34), namely,

$$\frac{1}{1+r}\mathbb{E}_{(K''|K')} \max\left[\pi K'' - B', 0\right].$$

We revisit Propositions 1-3 and ask how matters change when the option of costly equity finance is available.

First, note that since debt claims have priority over equity claims, the payoff to creditors conditional on a B' is the same regardless of whether or not there are equity contracts outstanding. Therefore, access to bond finance and the bond price schedules are exactly as described in Propositions 1-3. Second, because access to bond finance is costless, it is optimal for a firm to borrow as

[21] We can allow a firm to sell fractional equity contracts but given the fixed of participation, this will not be optimal.

much as possible and issue an equity contract only if the net benefit from promising the residual cash flow to equity investors is positive.[22]

Consider the case in Proposition 2. For this range of θ values all future cash flow is committed to bond investors and, so, the availability of equity finance is immaterial. Thus, Proposition 2 remains valid.

Next, consider Proposition 3. In this range of θ values neither firm can borrow, so equity finance is potentially relevant. The final period expected cash flow of a K'-variety firm is $(1-\phi)\pi K'$. If this firm sells an equity contract, its net gain is $\alpha(r)(1-\phi)\pi K' - \kappa$, where $\alpha(r) = [1/(1+r) - \beta]$ is the benefit of bringing a unit of cash flow from the final period to the current period at the risk-free rate. In keeping with the prior that external finance is more accessible for larger firms, we assume that

$$\alpha(r)(1-\phi) < \kappa < \alpha(r)2(1-\phi). \tag{35}$$

Then a 1-variety firm will choose not to access the equity market and $W(1,0) = (1-\phi)[\pi+\beta(1-\phi)\pi]$, the same as in Proposition 3. However, the gain to a 1-variety firm from acquiring a new variety changes. Now,

$$W(2,0) - W(1,0) = W(1,0) + (1-\phi)^2[\alpha(r)2(1-\phi)\pi - \kappa]. \tag{36}$$

The gain is the stand-alone value of a 1-variety firm plus the expected gain that ensues from equity market participation conditional on both varieties surviving in the current period. The presence of this additional gain term — the pure financial benefit from an acquisition — means that (i) a new variety might be purchased even if its success probability is lower than in a startup, i.e., $s^*(1,0) < \sigma$ (misallocation) and (ii) since a lower r raises the pure financial benefit ($\alpha'(r) < 0$), a lower r reduces the number of startup attempts and, therefore, startups.

Finally, consider the case in Proposition 1. In this range of θ values the 1-variety firm has no access to bond finance but the 2-variety firm can and does borrow π. Given (35), the 1-variety firm is still without external finance. The situation of a 2-variety firm is more interesting. It has costless access to the debt market, which allows it commit π units of cash flow to bond investors.

[22] *If* it is optimal to sell an equity contract, any debt level satisfying the default probability constraint is optimal and the capital structure of a participating firm is, within limits, indeterminate.

So, the uncommitted cash flow is only π units in the state in which *both* varieties survive next period. Since this state occurs with probability $(1-\phi)^2$ a 2-variety firm will sell an equity contract if $\kappa < \alpha(r)(1-\phi)^2\pi$. But in view of (35), this inequality is violated. Thus, a 2-variety firm will choose not to participate in the equity market and Proposition 1 remains valid in its entirety.

To summarize thus far: Given (35), equity markets are not accessed by any firm unless the default probability constraint is very tight. And when it is, it is the bigger firm that accesses the equity market. Furthermore, it remains the case that the financial benefit is higher when the interest rate is lower.

In a fully dynamic setup, though, equity market participation will become profitable for a firm with a large enough K because the benefits will outweigh the cost κ, even if the firm is able to commit some fraction of its cash flow to bond investors. So, with costly equity market participation, we would expect a new entrant to start out with no external finance, move to bond finance as it acquires another variety and go public if it achieves a certain size.

We note also that in a fully dynamic setup, there will be mix of private and public companies in the population of firms. This raises another interesting question: Does organizational form affect the idea purchase decision? The answer is yes, if we view a public company as controlled by equity investors with a discount factor of $1/(1+r)$. Generally speaking, a private company will value an acquisition less than a public one because it can access the high valuation of patient investors only by committing future cash flows to them, subject to the default probability constraint. The consequence is that, all else constant, a public company will be more eager to buy new ideas and misallocation is more likely to happen with a public company. Lower interest rates will still affect entry negatively.

5 Quantitative Analysis

We now turn to a quantitative exploration of the full model.

5.1 Parametrization

We set the model period to be a month. The model has (i) two market parameters, r and θ, (ii) two preference parameters, β and γ, (iii) four technological parameters, M, ϕ, σ and $K_{\max}$, and (iv) the distribution $F(s)$. We assume F is a truncated normal on $[0,1]$ with mean $\bar{s}$ and standard

Table 1:
Parameters Set Independently

Parameter	Value
r per annum	0.018
β (annual)	0.975
$1/\gamma$	1.3
M per annum	120
$K_{\max}$	65
$\bar{s}$	0.7

Notes: For explanations, see text.

deviation ν. With this assumption, there are 10 parameters to be assigned. Note that αL is not counted in this list as its value is essentially a choice of units and can be normalized to 1.

The parameters listed in Table 1 are set independently. The risk-free interest rate r is set to 1.80 percent per annum, which is the trend value of the annual average real return on 3-month Treasury bills in 1997.[23] The exact value of β is not important for the results, but it is important that firms be more impatient than lenders. We set β to 0.975. Based on Hall (2018), we set $1/\gamma = 1.3$.[24] The choice of M is, effectively, a normalization[25] and we set M to a numerically convenient value of 120 (i.e., 10 per month).

The two remaining parameters in Table 1, $\bar{s}$ and $K_{\max}$, are set with numerical tractability in mind. The distribution $F(s)$ is a truncation to $[0, 1]$ of a normal distribution with mean $\bar{s}$ and standard deviation ν. All else constant, an decrease in $\bar{s}$ decreases the fraction of new ideas that are successfully implemented and, so, decreases the measure of varieties in steady state. With M constant this results in an increase in ρ. A higher ρ and/or a higher $K_{\max}$ lead firms to encounter several new ideas per period, which is numerically cumbersome. To avoid this, we set $\bar{s}$ to 0.70 and

[23] The real Treasury bill rate is the annualized nominal interest rate on 3-month Treasury bills at the start of a year minus the CPI inflation over the previous year. The trend value in 1997 is the value of the HP trend ($\lambda = 100$) in 1997. The sample period used to compute the HP trend is $1978 - 2018$.

[24] Hall estimates Lerner indexes and their growth for US industries. For $2007 - 08$, the midpoint between 1997 and 2018, his Figure 6 suggests an aggregate Lerner index in the vicinity of 0.23, which corresponds to a $1/\gamma$ of 1.3.

[25] If M is changed by a factor λ, the new steady-state mass at any point (K, B) changes by a factor λ as well. So, $\mu(K, B)$ changes by a factor λ at all points and the new steady state measure of varieties changes by the factor λ. Therefore, by equation (24), the value of ρ is unaffected in the new steady state. Furthermore, even though a change in the measure of varieties changes $\pi(\cdot)$ by equation (8), in steady state a change in π acts like a change in units: all value functions and debt choices get appropriately scaled up or down, leaving acceptance and default probabilities unchanged.

Table 2:
Parameters Set Jointly

Description of Statistic	Data Target	Parameter	Value
Probability of default	0.01	θ	0.080
Survival rate of 1-yr-old firms	0.86	ϕ	0.195
Employment share of entrants	0.027	σ	0.622
Trend decline in empl share of entrants, 1997-2018	0.012	ν	0.077

Notes: All data and parameter values expressed in annualized terms. The data for the employment share of entrants and survival rate of 1-yr old firms are authors' calculations based on data from the U.S. Census Bureau's Business Dynamics Statistics (https://www.census.gov/ces/dataproducts/bds/data.html).

$K_{\max}$ to 65 to keep the numerics tractable. Small changes around these values do not affect model statistics much at all.[26]

The top panel of Table 2 reports the value of the remaining 4 parameters, which are set jointly to deliver realistic model statistics. Each row lists the statistic that is matched and the parameter value that is most directly determined by the match.

The first statistic listed is the default probability on debt. For this statistic, we use the bankruptcy rate for firms reported in Corbae and D'Erasmo (2021, Table 1) and match a default probability of 1 percent.[27] In the model, the parameter that most affects the default rate is θ and its matching value is 0.08.[28]

The next three statistics are trend values derived from the U.S. Census Bureau's Business Dynamics Statistics (BDS). These values are calculated for the period 1978-2018 using the Hodrick-Prescott filter for annual data (with $\lambda = 100$).

The trend value of the survival rate of 1-year old firms in 1997 is 86 percent. The parameter that most affects this survival rate is the product extinction probability ϕ, with the survival rate declining with increases in ϕ. The matching value of ϕ is 0.195.

[26] Given all other parameters, the steady state share of firms at $K_{\max}$ is very small (less than 10^{-5} percent).

[27] They report an overall bankruptcy rate of 0.96 percent for the period $1980 - 2014$. Once we take into account that only around 90 percent of firms carry debt, bankruptcy rate conditional on debt is 1.1 percent.

[28] Since θ is chosen to match a default rate of 1 percent, we might expect θ to be 0.01 but its value is higher. This is a consequence of $d(K', B')$ being a step function in B'. Because the probability of default jumps up discretely with increasing debt, there is no $B' \in \mathbb{B}$ (generically) for which $d(K', B')$ is *exactly* θ. Thus even if a firm is at its default constraint, its default probability will be strictly less than θ. For this reason, to match an average default probability of 1 percent, θ will have to exceed 1 percent.

The trend value of the share of employment of new firms in 1997 is 2.7 percent. In the model, we count a firm as an entrant if it was created in the current period or any time in the last 11 model periods and survived. The entrant's share of employment is the measure of varieties produced by entrants divided by the total measure of varieties at the end of the current period.[29] The parameters that most affect the entrant's share of employment is the success probability of startups, σ, and the standard deviation ν of s. A higher σ directly increases the probability that a new firm will be created and a higher ν increases the probability that an idea's s will fall below the relevant s^* and be implemented by a startup.

The parameter ν also affects the response of entrant employment share to a change in r. In the data, the trend value of entrant employment share fell by 1.2 percentage points (from 2.7 percent in 1997 to 1.5 percent in 2018). In Section 6 we argue that the decline in r over this period is a possible reason for this drop. Anticipating that discussion, we set the values of σ and ν to match the entrant employment share in 1997 and the trend decline in this share (the interest rate path we assumed to generate this decline is discussed in Section 6). The matching values of σ and ν are 0.617 and 0.077, respectively.

Finally, one variable that decision makers treat as a parameter is the equilibrium path of $\rho(X)$. In steady state, $\rho(X)$ is constant and if every idea became a new variety its value would equal ϕ (otherwise the measure of varieties will either grow or shrink). But since only a fraction of new ideas are successfully implemented, the steady-state ρ exceeds ϕ and its annualized value is 0.27.[30] We note that at the monthly value of steady state ρ (which is 0.026) firms with $39 \leq K \leq K_{\max}$ get one idea for sure and a second idea with some probability (see Appendix B for a description of how the decision problem changes when $\rho(X)K$ can exceed 1).

5.2 Model Mechanics

5.2.1 Leverage and Firm Size

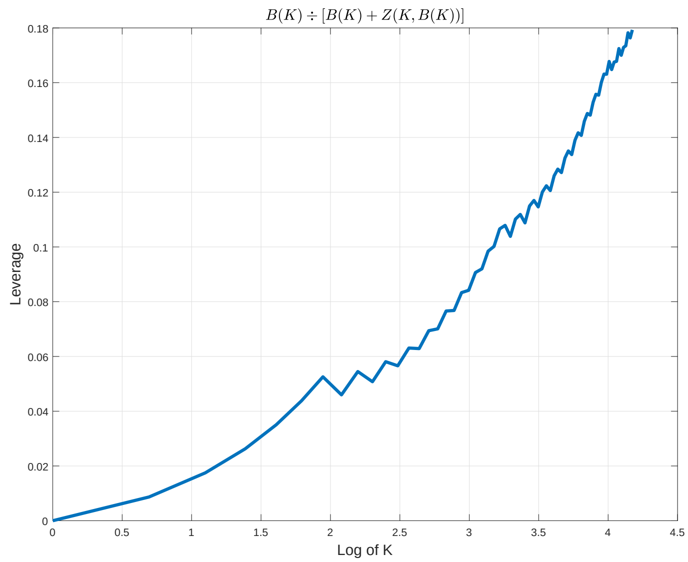
One property of the model that is of interest to us is the relationship between leverage and firm size. Figure 2 displays leverage, as defined in (19), against $\log(K)$ (the model analog of the size measure in Figure 1 and in the regressions reported in Appendix A). Although the relationship

[29] The share of employment of entrants is identical to the share of varieties of entrants because each variety employs the same measure of workers.

[30] If the average success probability of ideas implemented in existing firms exceeds σ, then $\rho \leq \phi/\sigma = 0.31$. The steady state equilibrium value of ρ is close to this upper bound.

is not strictly monotonic (more on this below), leverage is generally strongly increasing with firm size.

Figure 2:
Leverage and Firm Size



To understand the logic underlying this figure, assume that a firm can acquire at most one variety next period and, for compactness of notation, let $p^*(K', B')$ denote the probability of acquiring a variety, that is,

$$p^*(K', B') = [\rho(X')K' \int s\mathbb{1}_{\{s \geq s^*(K',B';X')\}} dF(s)].$$

Then, the probability of default on debt B' issued by a firm with K' varieties is

$$d(K', B') = p^*(K', B') \cdot \Pr\left[\pi K'' + G(K'') \ \leq B' \ |K' + 1\right] \ + \\ [1 - p^*(K', B')] \cdot \Pr\left[\pi K'' + G(K'') \leq B' \ |K'\right].$$

For this discussion, we will make two assumptions, both of which hold for the calibrated model: First, that the total resources available to a firm — both internally and externally — are increasing in firm size, i.e., $\pi\widetilde{K} + G^*(\widetilde{K})$ is increasing in $\widetilde{K}$. And, second, that the probability of acquiring a new variety is decreasing in legacy debt, i.e., $p^*(K, B)$ is decreasing in B — which is just the debt overhang effect discussed in Section 4.

Since $\pi\widetilde{K}+G^{*}(\widetilde{K})$ is increasing in $\widetilde{K}$, the two Pr[·] terms are the probabilities of a *tail event*: the event that K'' falls below a debt-dependent threshold $\bar{K}(B')$. Then, it is intuitive that the first Pr[·] term must be less than the second Pr[·] term: It is less likely that the number of surviving varieties will fall below $\bar{K}(B')$ if the firm starts out with more varieties. From this it follows that, given B', the probability of default declines with K'. Furthermore, the presence of the debt overhang effect is sufficient to ensure that, holding K' fixed, the probability of default rises with B'. Short proofs of these facts are given in Appendix D. The key implication of these facts is that the maximum amount a firm can borrow without violating the default constraint is increasing in K'. In other words, a firm's debt capacity is increasing in firm size.

For another way to understand why debt capacity is increasing in firm size, observe that the $d^{*}(K',B')$ can be expressed as

$$d^{*}(K',B') = p^{*}(K',B')\cdot \Pr\left[\frac{\pi K''+G(K'')}{\pi K'} \leq \frac{B'}{\pi K'}\;|K'+1\right] + \\ [1-p^{*}(K',B')]\cdot \Pr\left[\frac{\pi K''+G(K'')}{\pi K'} \leq \frac{B'}{\pi K'}\;|K'\right]. \tag{37}$$

If $\pi\widetilde{K}+G(\widetilde{K})$ changes approximately linearly with $\pi\widetilde{K}$ then the term $[\pi K''+G(K'')]/\pi K'$ becomes approximately proportional to K''/K'. The mean and variance of this random variable is proportional to $(1-\phi)$ and $(1-\phi)\phi/K'$, respectively. Therefore, as K' increases the mean of K''/K' is unchanged, but its variance shrinks, which directly implies that tail probabilities shrink. Consequently, probability of default given B' declines with K' and, given K', it increases with B'. As the ratio K''/K' is the realized (gross) rate of growth of sales, (37) also explains why the volatility of sales *growth* is a key determinant of default probabilities.

The above argument explains why a larger firm can borrow more. We turn now to whether it would would *want* to do so. Consider a B' at which the default probability constraint is not binding. Then, the firm can issue some small amount of debt Δ and consume an additional $q(K',B'+\Delta;X)\Delta$. In return, conditional on not defaulting, it gives up an additional Δ units of its (real) cash flow next period and, so, in expectation it gives up $\beta[1-d(K',B'+\Delta;X)]\Delta$ in utility terms (in default dividends are already zero and, so, the firm does not give up anything more). Since $q(K',B'+\Delta;X) \geq [1-d(K',B'+\Delta;X)]/(1+r)$ and $\beta(1+r)<1$, there is a gain from issuing additional debt.

But there are two countervailing forces as well. First, as shown for the two-period model for legacy debt, a higher B' reduces the probability of the firm acquiring a new variety next period via the debt overhang effect. Thus, accumulation of debt extracts a price in terms of future cash flow forgone. And, second, a higher B' may lead to a binding, or more binding, non-negativity constraint on dividends along the repayment branch: legacy debts may force a firm to issue more debt than it otherwise would, just to meet its obligations to creditors. These countervailing forces, however, turn out to be quantitatively weak (more on this below). The reason is that the upper bound on default probabilities, θ, is calibrated to be quite low, so firms cannot lever up to the point where these countervailing forces can exert significant effects. The end result is that the utility gain from bringing consumption forward in time is the dominant force, and firms expand their debt to the point where the default probability constraint binds. But this is the same as saying that the optimal debt level of a firm is the level that attains $G(K')$.[31] An implication is that $B(K', B, X)$ is actually independent of B and just depends on K' (and X).

Finally, note that even though firms borrow up to the maximum feasible level, generically, $d^*(K', B^*(K)) < \theta < d^*(K', B^*(K)+\Delta)$, where Δ is the step-size of the B grid (see the discussion in footnote 27). Consequently, it is often the case that $\theta < d^*(K'+1, B^*(K)+\Delta) < d^*(K', B^*(K)+\Delta)$. In other words, even though default probabilities decline with increase in firm size, the decline may be too modest to expand the borrowing capacity of the firm. In such situations, equilibrium *leverage* will fall. This accounts for the small dips we see in Figure 2. Of course, with a big enough increase in firm size, debt capacity expands and equilibrium leverage goes up (as seen in the Figure).

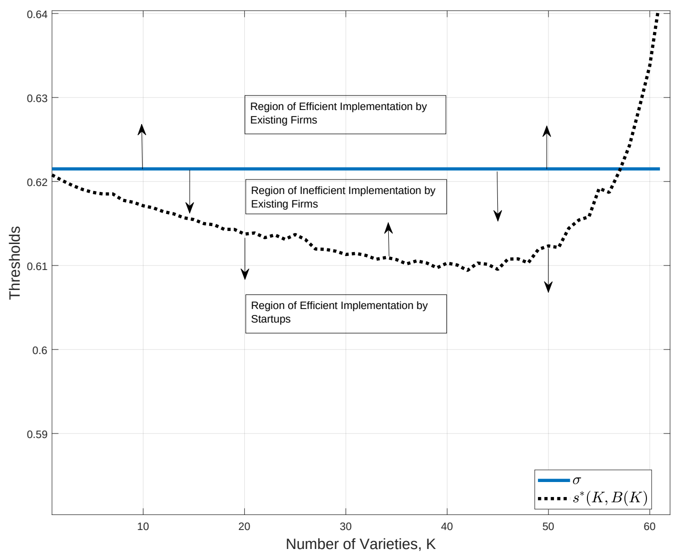
In summary, the generally positive relationship between leverage and firm size arises for essentially the same reasons as in the two-period model: Maximum feasible leverage is increasing in firm size because the growth rate of sales is less volatile for a larger firm and impatience induces business owners to borrow to the point where the default probability constraint binds.

5.2.2 Larger Firms Are More Willing to Buy New Ideas

The message of Figure 2 is that larger firms are able to borrow a greater fraction of their firm value. From our discussion in the context of the two-period model, we should expect new ideas to be more valuable to larger firms. This is another property that is of interest to us. Figure 3 shows that it holds for full model, provided K is not too close to the upper bound $K_{\max}$.

[31] Thus, along the repayment branches, a firm chooses to borrow an amount that its creditors would force it borrow if it were in default.

Figure 3:
Firm Size and the Willingness to Buy New Ideas



The dotted line in the figure plots the equilibrium threshold higher of $s^*(K, B^*(K))$ against K and σ (the success probability of an idea in a startup). The threshold declines monotonically with K, as expected, but only up to $K = 42$. Beyond 42, the upper bound $K_{\max}$ begins to exert a countervailing force. At $K_{\max}$ a firm must reject all ideas no matter how high the success probability, i.e, $s^*(K_{\max}, \cdot) = 1$ (not shown in the figure). As K approaches $K_{\max}$, the option value of waiting for an idea with a higher s rises and, so, s^* rises as well.

A couple of implications follow: First, up to a firm size of 42, a larger firm is more likely to buy new ideas than a smaller firm. Second, for a given K, if s falls between the solid line and the dotted line, there is misallocation: The idea is implemented in the firm where the success probability is lower than in a startup. There is a possibility of misallocation of this type for all firm sizes except the top nine firm sizes. And, up until $K = 42$, the possibility of misallocation is at least as high for larger firms as compared to smaller ones. For the top nine firm sizes there is also a possibility of misallocation but in the other direction: A firm may get an opportunity to buy an idea that has a higher success probability if implemented in the firm rather than a startup, but it will reject it because of an approaching or binding constraint on firm size.

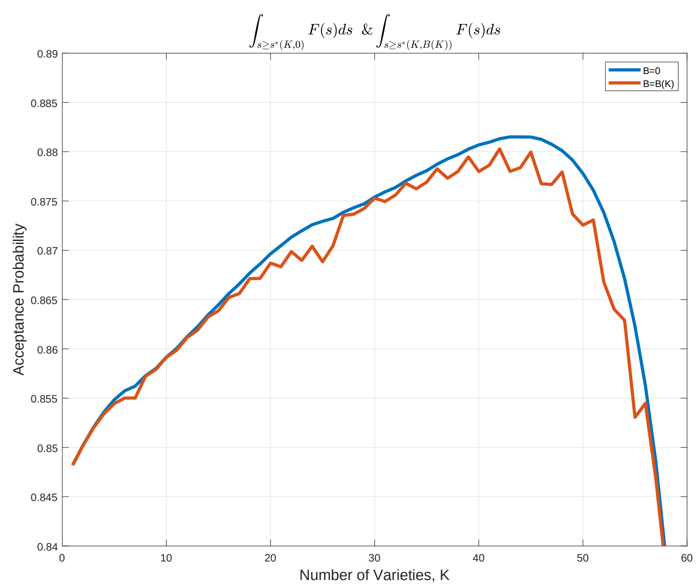
As we noted earlier, our model abstracts from accumulation of capital. If capital accumulation were to be permitted, then, because larger firms can lever up more, they would be able to invest in physical and customer capital at a faster rate relative to startups. This force is missing in

Cooley and Quadrini (2001) as they do not allow existing companies to buy new ideas. Although permitting capital accumulation might lead to interesting dynamic implications, adding another state variable into the model is a challenge and we abstract from it.

5.2.3 Debt Overhang

In the simple two-period model of Section 4, we showed that legacy debt made the firm more choosy about the ideas it buys. Figure 4 confirms this debt overhang effect in the full model. It plots the acceptance probability of a new idea presented to a firm of size K when it starts the period with the equilibrium debt level $B^*(K)$, i.e., it plots the quantity $\int_{s \geq s^*(K,B^*(K))} F(s)ds$, and it plots the acceptance probability of a firm of size K that arrives into the period with zero debt, i.e., the quantity $\int_{s \geq s^*(K,0)} F(s)ds$. Since a firm with $K = 1$ cannot borrow, $s^*(1, B(1)) = s^*(1, 0)$ and since a firm with $K = K_{\max}$ does not accept any new ideas, $s^*(K_{\max}, B^*(K_{\max})) = s^*(K_{\max}, 0) = 1$, Otherwise, the latter probability is always above the former, although the gap is quite small.

Figure 4:
Decline in Probability of Acceptance with Debt



The intuition for this debt overhang effect is similar to the logic of the two-period model. Given an inherited debt level B, there is a level of $\bar{K}(B)$ below which default is triggered. If the firm acquires a new variety, $\bar{K}(B)$ does not change and, so, the probability of default declines. In addition, conditional on default (that is, ending with a $K' \leq \bar{K}(B)$), the expected number of varieties is higher if the firm acquires a new variety, and this increases the recovery on the defaulted debt. On both counts, some portion of the cash flow of the new variety is captured by the firm's existing creditors, thereby blunting the firm's desire to acquire a new variety.

Related to the debt overhang effect is the possibility of a binding nonnegativity constraint on dividends. To pay off creditors the owners might prefer to take a current loss (negative dividends) and curtail new borrowing rather than issue additional debt and exacerbate the debt overhang problem. In this situation, the nonnegativity constraint on dividends will bind. However, this never happens in the calibrated model: The amount borrowed along repayment branches is the same as what the firm would borrow if at the time of its borrowing decision its inherited debt is set to zero. Again, this aspect of the equilibrium is a reflection, ultimately, of the relatively tight constraint on default probability.

5.3 Untargeted Empirical Properties

5.3.1 Leverage and Firm Size

How does the positive response of leverage to K compare with the response of leverage to firm size in the data? We computed, for each (K_n, B_n) in the steady state distribution, the pair $\{K_n, B_n/[B_n + Z(K_n, B_n)]\}$. Treating each pair as an observation (weighted by their measure), we regressed $B_n/[B_n + Z(K_n, B_n)]$ against $\log(K_n)$. The coefficient from this regression is 0.024, which is in range of the values reported in the top row of Tables 7 and 8 and in Dinlersoz, Kalemli-Ozcan, Hyatt, and Penciakova (2019).

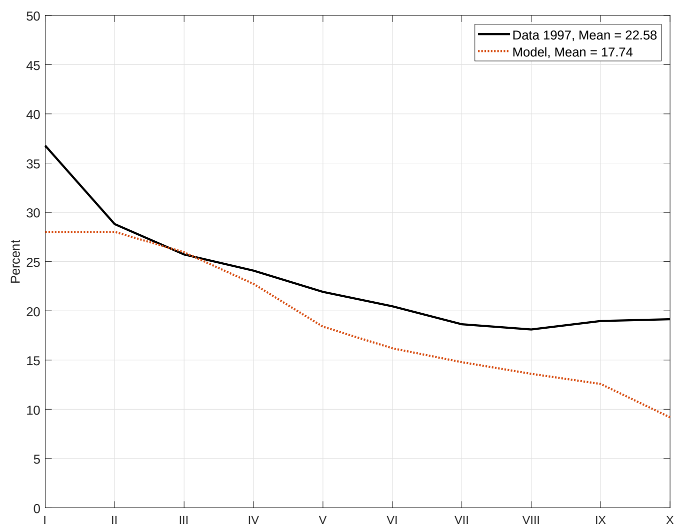
We can also compare average leverage in the data and in the model. In the data, average leverage for the case where leverage is measured as the ratio of debt to market value of assets (the measure of leverage used in Figure 1) is 0.094 and the median leverage is 0.076. In the model, mean leverage is only 0.016, almost an order of magnitude lower. We suspect that this is because debt in the model is unsecured debt, whereas a significant portion of business debts is secured against collateral.

5.3.2 Volatility and Firm Size

The best evidence regarding firm-level volatility is the firm-level job creation and job destruction rates reported in the Census's Business Dynamics Statistics (henceforth BDS) database. For the measure of firm-level employment growth volatility, we use the 1997 reallocation rates excluding births in each BDS size category.[32] To construct the model analog of this measure we follow the same procedure, treating each K as a size category. Once this is done, we distribute the total measure of firms in order of increasing size into ten bins with the same employment share as each of the BDS size categories and compute the average employment volatility measure for each "BDS" bin.

Figure 5 displays the results. Both data and model display a negative relationship between firm-level volatility in employment growth and size. But the decline in volatility in the model is sharper than in the data. To put the figure in perspective, note that for a standard CRS technology

Figure 5:
Firm Volatility and Firm Size



N.B.: The data size categories correspond to employment levels of 1-4, 5-9, 20-99, 100-499, 500-999, 1000-2499, 2500-4999, 5000-9999, 10000+. The model size categories are constructed as described in the text.

with firm-specific multiplicative technology shock A, the growth rate of a firm is a draw from the process for the growth rate of A. Since the process for A is generally assumed to be independent of firm size, the growth process for firms is predicted to be independent of firm size as well. This prediction is at variance with the figure.

[32]Specifically, for each size bin we use the formula

$$\frac{\text{Job\_Creation\_Continuers in t} + \text{Job\_Destruction in t} - |(\text{Job\_Creation\_Continuers in t} - \text{Job\_Destruction in t})|}{\text{DENOM}},$$

where DENOM is the average of employment in the bin for times t and t-1 (the denominator recommended by BDS). See variable definitions in BDS documentation for more detailed explanation of these measures.

The volatility of employment growth does decline with firm size in our model. The decline is a manifestation of the decline in the variance of K''/K' with K'. Evidently, the decline in the model is sharper than in the data. This could be a measurement issue: in the data, large firms have multiple establishments and reallocation of employment across these establishments are counted. In the model we treat each firm as having a single establishment (producing multiple products) so all within-firm acquisitions and extinctions of product varieties are netted out. Another source of the discrepancy might be a missing force that causes the employment volatility of large firms to decay slower than $1/K$ (more on this below).

5.3.3 Firm Size Distribution

We turn now to the firm-size distribution in the data versus the model. In the data, the firm size distribution is reported for the BDS employment categories of Figure 5. In this instance, it is more informative to transform the BDS size categories into categories comparable to those in the model. Specifically, we associated with each BDS size category the average firm size for the category as a multiple of the average firm size of the first (smallest-size) category. This gave us size categories (to the nearest integer) of 1,3,6,17,83,263,565,1088,1079, and 11104.[33] The model's size categories are just $1, 2, 3 \ldots, 65$.

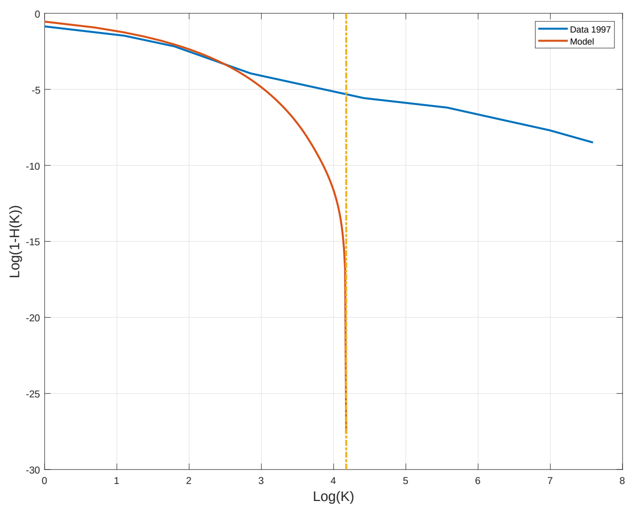
Let $H(K)$ denote fraction of firms of size K or less. Figure 6 plots $\log(1 - H(K))$ against $\log(K)$ for the data and the model. For the data, the plot is downward sloping and almost linear, reflecting a Pareto law. The situation is very different for the model: the model plot "hugs" the data plot for up to a firm size of 10 (i.e. 2.3 in log terms), but then begins to fall precipitously. This is to be expected: In the model, the largest firm is only 65 times the size of the smallest firm and the fraction of firms of that size is very small. Thus $[1 - H(K)]$ begins to approach zero rapidly as K approaches 65. The main consequence is that the model downweights the frequency of large (especially very large) firms and upweights that of smaller firms.

We suspect that this gap results, in part, from the fact that the model does not allow a firm to become large *via* a variety that is particularly desirable to consumers or a technology that is particularly cost effective. If we, in some fashion, managed to get very large firms (say, a firm with 11,000 varieties) then we would create a huge discrepancy with respect to the size and volatility relationship displayed in Figure 5 because such a large firm will have virtually predictable growth.

[33] The average firm size in the eighth and ninth categories appears anomalous, but such reversals are possible.

To get both the size distribution and the volatility-size relationship right, we would need to add variety-specific heterogeneity, which is a nontrivial task.[34]

Figure 6:
Firm Size Distribution



6 The Risk-Free Rate and Firm Dynamics

In this section, we examine the implications of a lower real interest rate on the steady state equilibrium of our economy. The motivation for this investigation is the well-known decline in the real interest rates and startup rates over the past several decades. Figure 7 shows the secular movement in both the startup rate and the real interest rate over the period 1978-2015.[35] Importantly, as shown in Figure 8, the share of corporate profits in GDP has risen strongly since the late 1990s.[36]

The fact that the decline in startup rates continued, even accelerated, in the face of rising profits is puzzling. But in our model, there is a new margin that effects the entry rate, namely, the choice of organization within which a new idea is implemented. As explained in Section 4, a decline in r causes more ideas to be implemented in existing firms, leading to lower startup attempts and fewer startups. Interestingly, Akcigit and Ates (2019, Figure 9, p. 45) document that the share of new patent applications (in total applications each year) that are registered to the top 1 percent

[34]We want to highlight that the gap between the firm size distribution in the model and the data cannot be fixed by a *firm-level* productivity shock because such shocks would have the implication that when a large (and efficient) firm acquires a variety, the variety will automatically inherit the productivity of the acquiring firm. While there might be some truth to this, it is implausible to think that any product that, say, Apple buys will have the success of an iPhone.

[35]The entry rate is available since 1978 and is from the U.S. Census Bureau's Business Dynamics Statistics database. For the definition of the real interest rate, see footnote 23.

[36]The share of corporate profits is the ratio of annual Corporate Profits after Tax (without IVA and CCAdj) to annual Gross Domestic Product published by the U.S. Bureau of Economic Analysis.

Figure 7:
Entry Rate, Entrant Empl Share & Real Interest Rate, 1978-2018

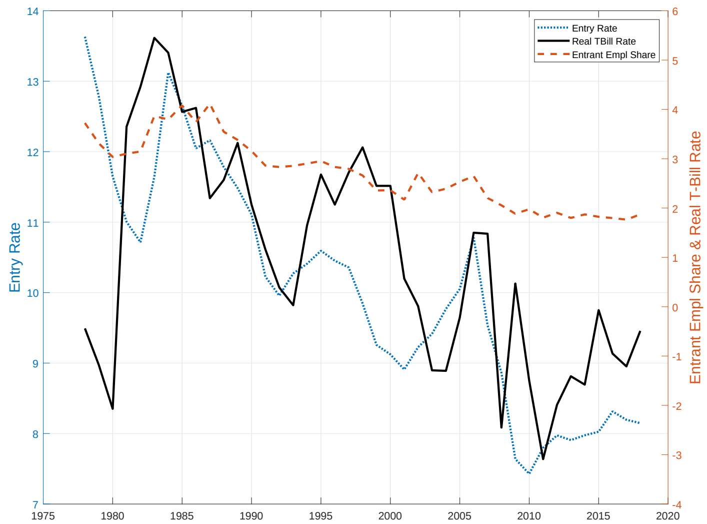
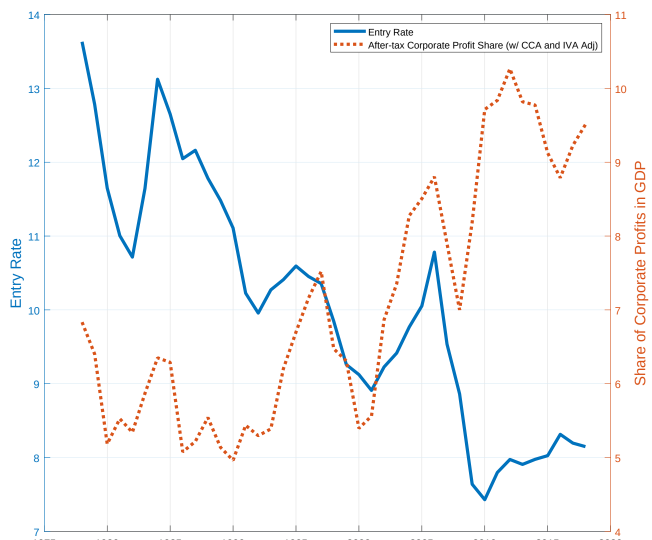
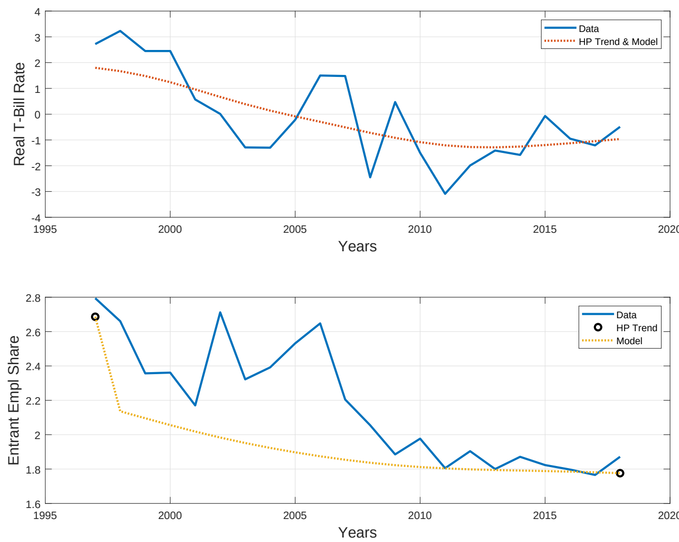


Figure 8:
Entry Rates and Share of Corporate Profits, 1978-2018



of patent holders has risen by around 15 percentage points since the early 1980s, with the bulk of the rise occurring since the mid-1990s. Simultaneously, the share of new patent applications registered to first-time patenters has declined since the early 1980s, with the bulk of the decline again occurring since the mid-1990s. At a broad level, these patenting patterns are consistent with our model implication that a lower interest rate results in the implementation of new ideas moving out of startups and into large firms.

Figure 9:
Real Interest Rates & Entrant Employment Share

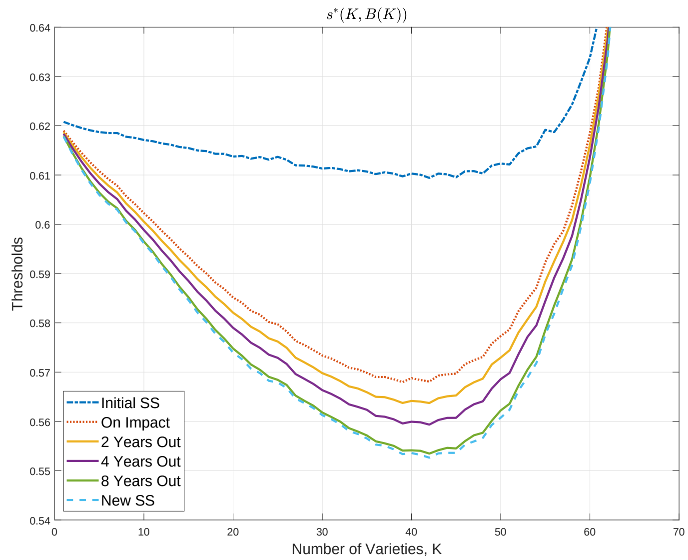


It is to quantitatively investigate this possibility that we chose parameter values to reproduce employment share of entrants in 1997 *and* the trend decline in this share between 1997 and 2018. Thus all the model–implied magnitudes reported below assume that the decline in the entrant employment share that results from the drop in the risk-free rate is equal to the actual drop in the entrant employment share over this period.

We now explain how we match the trend decline in the entrant employment share. First, we assume that the economy is in steady state with r at its 1997 trend value of 1.80 percent (the initial setting of r). Then, agents learn that r will follow the generally declining trend path shown in the top panel of Figure 9, with r expected to remain constant -1.0 percent (its 2018 trend value) beyond 2018. We solved for the transition following this unanticipated shock to expectations and chose ν in conjunction with σ (and other parameters) to ensure that the decline in entrant employment share at the end of 22 years (i.e., at the end of 12×22 model periods) matches the data.

The bottom panel of Figure 9 shows the result of this matching exercise. In the data, the trend path of entrant employment share declines from 2.7 percent to 1.5 percent. The calibration makes the model exactly match the start and end points of the trend line (denoted by the black circles). The model predicts a sharp drop in the entrant share within the first year of the shock and a slow decline in the ensuing years (more on this below).

Figure 10:
$s^*(K, B(K))$ Schedule Along the Transition Path



To understand how the declining path of r affects equilibrium outcomes, recall that a firm makes two decisions each period: it chooses the threshold $s^*(K, B)$, and it chooses B' given K' and B. Turning first to the latter, a lower r increases the revenue from bond sales and, all else constant, increases $\bar{B}(K', B)$ (see (13)). This allows the firm to expand the amount of debt it can issue without altering default probabilities. This leverage effect, however, is very small on impact because the change in r on impact is minuscule; also, even when r falls measurably in later periods, the impact on leverage is not large.

In contrast, the effect on s^* of a change in the path of r is more substantial. Because the owners are forward looking, they recognize that lenders are willing to advance more funds against the cash flow of a product *every* period until product extinction. Thus, s^* declines on impact and, the decline is greatest on impact. As shown in Figure 10, thresholds continue to decline as time progresses, but in 8 years the schedule is virtually at its new steady-state configuration.

We now turn to the impact of the falling interest rate path on entry and business concentration.

6.1 Impact of Declining Interest Rates on Entry

Figure 11:
Impact Declining Risk-free Rate on Entry

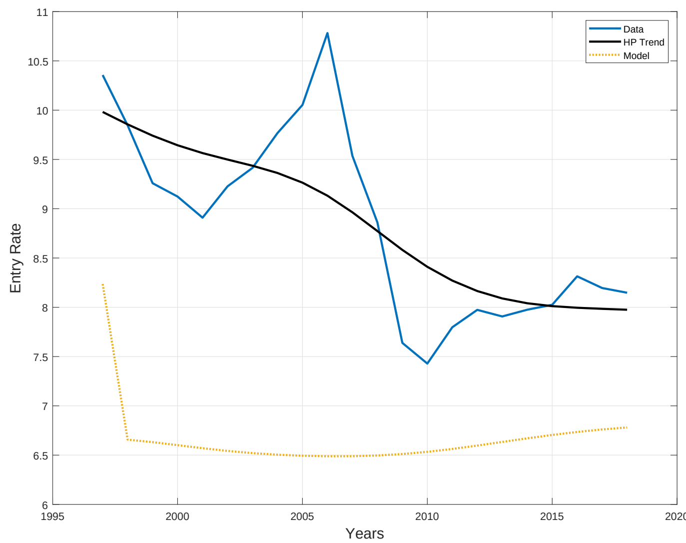


Figure 11 shows the model implied path of the entry rate. In the data, the entry trend declines 20 percent (from 10 percent to 8 percent). In the model, the entry rate declines 18 percent (from 8.3 percent to 6.8 percent). Since the decline in the s threshold is largest on impact, the induced decline in the startup rate is biggest on impact. After this initial drop, entry rates decline more gradually as s thresholds shift down gradually. Toward the end of the sample period, entry rates begin to rise in the data and the same tendency is also seen in the model-implied path of the entry rate. In the model, this small lift in entry rates is due to the gradual decline in the total measure of firms: when the measure of new entrants is not changing much toward the end of the sample period, the decline in the measure of firms leads to a rise in the entry rate.

The model's dynamic path of the entry, however, does not match the *gradual* nature of the observed decline in entry rates. In the model, decision makers anticipate the decline in r and, so, react to it on impact. In reality, the decline in r was probably not anticipated to the degree it is in the model. Dealing with this discrepancy in some way would likely mitigate the outsized impact effect on the s schedule and generate a more gradual decline in entry.

The model also predicts a lower entry rate relative to the data. This can be traced back to the fact that the model underweights the frequency of large firms. To see this, observe that the entrant share of employment can be expressed as

$$\frac{\text{Avg. size of entrants}}{\text{Avg. firm size}} \times \frac{\text{Measure of entrants}}{\text{Measure of firms}}$$

Since the model underweights the frequency of large firms, the first ratio is larger in the model than in the data. Since the model matches the employment share of entrants at the start and at the end of the sample by construction, the second ratio must be smaller in the model at the start and at the end of the sample. But the second ratio is the entry rate, which accounts for why model-implied entry rates are systematically below the entry rates in data.

6.2 Impact of Declining Interest Rates on Business Concentration

In the model, the decline in interest rates also has the potential to affect business concentration because the decline in the entry rate is accompanied by faster growth of existing businesses. Autor, Dorn, Katz, Patterson, and van Reenen (2017) document that the share of sales of the top 4 firms in 6 major industries has risen since the mid-to-late 1990s. In four out of the six industries, the share of sales in the late '90s ranged from 20 to 30 percent, and, in 2012, these shares had risen by 3 to 11 percentage points (Figure 1, p. 182). Since the share of sales accounted by any finite collection of firms in the model is zero, we examined, instead, the share of sales accounted by the *measure* of firms that made up the top 5 percent of all firms in the initial steady state.[37] As Table 3 shows, these firms accounted for 24 percent of sales (output), comparable to the share accounted for by the top 4 firms in the data. By 2012 (i.e., by $T = 16 \times 12$ model periods), the share of sales accounted for by the same *measure* of largest firms rises by 4 percentage point. This increase is less dramatic than the data but still significant. As further points of comparison, Table 3 also reports the rise in the sales share of firms in the top 3 percent and top 15 percent.

[37] Specifically, we used $\mu(K, B)$ to determine the measure of firms for each K and, then, starting with the firms with the largest number of varieties, we included firms with progressively fewer varieties until 5 percent of the total measure of firms was included.

Table 3:
Effects of Declining Interest Rates on Business Concentration

	Meas. of firms	Share of Output in %	
		Initial SS	Low r 2012
Top 5 percent by size (K)	5.84	24.1	28.4
Top 3 percent by size	3.50	16.9	20.1
Top 10 percent by size	11.67	37.8	43.5

6.3 Other Effects of Declining Interest Rates

Table 4 reports some of the other equilibrium effects of a declining path of interest rates. There is a modest increase in the responsiveness of leverage to sales, driven by the shift toward larger firms. The slight convexity of the plot in Figure 2 implies that a shift in the distribution toward larger firm size would increase the measured response of leverage to firm size.

There is a modest increase in the bankruptcy rate in the low interest rate economy, which is also the result of the shift in the distribution toward larger firms.[38] The actual change in the business bankruptcy filing rate over this period is in the opposite direction and quite significant: it fell from 1.1 percent in 1997 to 0.45 percent in 2018.[39] Evidently, our model sheds no light on this decline.

There is also a decline in the measure of varieties along the transition path. This is a consequence of the increased misallocation that results from the shift down in s^* schedule: The additional ideas purchased as a result of this shift have a *lower* success probability than the success probability of a startup. Consequently, a smaller fraction of the arriving ideas are, on average, successfully turned into new varieties and the equilibrium measure of varieties falls. An implication is that the decline in interest rates has led to a loss in welfare that comes from consuming fewer varieties. A flow measure of this loss, expressed as a percentage of annual real output in 2018, is $[(1/\gamma)-1]\times[dN'/N']\times 100 = 0.041$, or, about 4 basis points. In comparison to the welfare gains/losses typically found in macro models, this is not negligible. But its magnitude is probably quite sensitive to modelling details.

[38] A small firm's default probability is more sensitive to leverage (i.e., there are bigger upward jumps in default probability as leverage increases) and, hence, its default probability tends to be further away from θ as compared to a bigger firm. Consequently, as the fraction of small firms shrinks, mean default probability can go up.

[39] Administrative Office of U.S. Courts (Bankruptcy) recorded 53,993 and 23,443 business bankruptcy filings in 1997 and 2018, respectively, and for those years the BDS dataset recorded 4,781,344 and 5,252,110 businesses, respectively.

One that comes to mind immediately is that even a small positive impact of a lower r on M might eliminate or reverse the welfare loss.

Table 4:
Other Effects of Declining Interest Rates

Statistic	Initial SS	Low r 2018
Response of leverage to sales	0.024	0.025
Frequency of bankruptcies in %	1.00	1.16
Measure of varieties	387.13	386.68

Table 5 reports the model's implications for mean survival rates (of all existing firms) and mean employment growth rates, and the mean employment growth volatility and compares them to the data.

In the data, mean survival rates and employment growth rates have increased over the observation period. The same is true for the model, but in percentage terms the model increases are somewhat lower. The reason these rates rise in the model is the decline in s^* schedule: Existing firms are more likely to absorb varieties, which means they are more likely to grow and less likely to exit.

The mean volatility of employment growth, where employment growth volatility is measured as in Figure 5, falls over this period. The same is true in the model. In this instance, though, the decline in the data is more pronounced than in the model: The trend mean employment growth volatility declined around 16 percent, while it declines only about 6 percent in the model. The model decline occurs because of the shift toward larger firms.

Table 5:
Effects on Survival Rates, Empl Gr. & Empl Gr. Volatility

Statistics (all in %)	Data 1997	Initial SS	Low r 2018	Data 2018
Survival rate of existing firms	91.02	91.76	93.22	92.85
Empl. growth rate (gross) of existing firms	99.27	97.31	98.22	100.45
Mean employment growth volatility	21.29	17.99	16.92	17.80

The data are from the U.S. Census Bureau's Business Dynamics Statistics database (https://www.census.gov/ces/dataproducts/bds/data.html). Each data point is the HP trend (λ=100) for the indicated year, where the trend is computed for the sample 1978-2018.

7 Conclusion

We presented a model in which firms manage collections of product varieties. The arrival into the economy of new varieties and the extinction of existing varieties are random events. Since firms manage collections of varieties, the random process entry and exit of product varieties induces a stochastic process for the entry, growth, and exit of firms. A firm's access to capital markets plays a key role in firm dynamics. Because larger firms have less volatile cash flows, they can borrow more, which generates a positive association between firm size and leverage, consistent with the evidence. As a result of this positive relationship, a lower risk-free rate makes it more profitable for larger firms to buy out ideas for new varieties from startups and, so, causes a decline in the startup rate and greater concentration of sales among top firms. Our paper, thus, connects the decline in the startup rate and the rise in business concentration since the mid-1990s to the decline in the risk-free rate over this same period.

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Appendix A

Table 6: Variable Names and Description

Variable	Description
SALES/TURNOVER (Net)	Sales
LT	Total Liabilities
DLC	Debt in Current Liabilities - Total
DLTT	Long-Term Debt - Total
CSHO	Common Shares Outstanding
PRCC_F	Price Close - Annual Fiscal
CHE	Cash and Short-Term Investments
PPENT	Property, Plant and Equipment - Total (Net)
EMP	Employees
CEQ	Common/Ordinary Equity - Total
OIBDP	Operating Income Before Depreciation
AT	Assets - Total
lnSale	ln(SALES/TURNOVER (Net))
Book_Val	AT
Mkt_Val	AT − CEQ + (CHSO×PRCC_F)
Mkt Leverage Ratio I	(DLC + DLTT − CHE)/Mkt_Val
Mkt Leverage Ratio II	(LT − CHE)/Mkt_Val
Book Leverage Ratio I	(DLC + DLTT − CHE)/Book_Val
Book Leverage Ratio II	(LT − CHE)/Book_Val
Cap_Ratio	PPENT/Mkt_Val or PPENT/Book_Val
Profit_Ratio	OIBDP/Mkt_Val or OIBDP/Book_Val

Our constructions follow the norms in the finance literature. The book value of a firm is total assets (AT) reported in COMPUSTAT and market value is book value excluding common equity (CEQ) plus the market value of its common equity outstanding (CHSO × PRCC_F). We use two measures of a firm's leverage ratio: In the first, we include only liabilities that arise as a result of a firm's active borrowing net of cash (DLC + DLTT - CHE); in the second, we use total liabilities net of cash (LT - CHE). For our measures of leverage, we divide these two net liability measures by either the market value of assets or the book value.[40] For use as controls, we define the tangible capital ratio of a firm (capital ratio for short) as the value of its tangible capital (PPENT) to asset

[40]The resulting leverage measures are commonly used in the literature. For instance, book leverage based on debt is used in Rajan and Zingales (1995), Hennessy and Whited (2005), and Lemmon, Roberts, and Zender (2008), among many others; book leverage based on total liabilities is used in Baker and Wurgler (2002); market leverage based on debt is used in Whited (1992), Rajan and Zingales (1995), and Strebulaev and Yang (2013); and market leverage based on total liabilities is used in Baker and Wurgler (2002) and Michaels, Page, and Whited (2019). See Frank and Goyal (2009) for a discussion of the pros and cons of these different leverage measures.

value (market or book) and the profit ratio as value of its operating income before depreciation (OIBDP) to asset value (market or book).

Table 7:
Leverage & Firm Size (Panel)

	Market Value		Book Value	
Dependent Var	Lev I	Lev II	Lev I	Lev II
lnSale	0.035	0.039	0.038	0.040
	(23.94)	(23.29)	(23.65)	(23.34)
Cap_Ratio	0.287	0.423	0.589	0.522
	(23.91)	(28.83)	(49.27)	(41.49)
Profit_Ratio	-0.035	-0.146	-0.011	-0.010
	(-3.70)	(-10.26)	(-1.97)	(-1.83)
Firm Fixed Effect	Yes	Yes	Yes	Yes
Year Fixed Effect	Yes	Yes	Yes	Yes
Num of Obs	222,596	223,397	259,589	251,447
Num of Groups	22,554	22,583	25,370	25,005
R^2	0.14	0.08	0.17	0.08

Source: Authors' calculations using COMPUSTAT (1978-2015), Compustat data from S&P Global Market Intelligence, retrieved from Wharton Research Data Service (WRDS) on 12-15-21. For each year, only firms (i) reporting in U.S. dollars, (ii) with book value, market value and sales of at least \$1 million in 2015 (the GDP deflator is used to determine the equivalent nominal cutoff for earlier years), (iii) with nonnegative debt and cash holdings, and (iv) leverage between -1 and $+1$ are included.

Tables 7 and 8 report the results for panel and cross-sectional regressions, respectively (t-statistics in parentheses).[41] In all regressions, the regressor of interest is the logarithm of sales (indicator of firm size). All regressions include the capital ratio and the profit ratio as well as year dummies. The panel regressions include a firm dummy and the cross-sectional regressions include a (sub) industry dummy.

Turning first to Table 7, the coefficient on log sales is positive, highly statistically significant, and roughly the same value across the four regressions. The estimates imply that every doubling of firm size increases leverage by 2.4 to 2.6 percentage points. Turning to Table 8, the coefficient on log sales is again highly statistically significant but generally somewhat lower than the coefficients reported in Table 7.

[41]In all regressions, the denominators in Cap_Ratio and Profit_Ratio use Mkt_Val (Book_Val) if the dependent variable is market (book) leverage.

Table 8:
Leverage & Firm Size (Cross-Section)

	Market Value		Book Value	
Dependent Var	Lev I	Lev II	Lev I	Lev II
lnSale	0.014	0.025	0.022	0.038
	(59.28)	(91.06)	(76.40)	(125.72)
Cap_Ratio	0.292	0.385	0.457	0.341
	(115.16)	(133.45)	(132.39)	(93.93)
Profit_Ratio	0.003	-0.128	-0.011	-0.012
	(0.75)	(-30.71)	(-18.73)	(-20.48)
Subindustry Fixed Effect	Yes	Yes	Yes	Yes
Year Fixed Effect	Yes	Yes	Yes	Yes
Num of Obs	217,328	218,117	252,329	244,438
R^2	0.28	0.46	0.33	0.40

N.B. See notes to Table 7.

Appendix B

This Appendix describes the general choice problem wherein ρK can be any positive real number (to reduce notational burden, we drop the dependence of ρ on X). Such a firm gets the opportunity to buy $\lfloor \rho K \rfloor$ ideas for sure and the opportunity to buy the $\lfloor \rho K \rfloor + 1^{th}$ idea with probability $\rho K - \lfloor \rho K \rfloor$. At each purchase node, the firm knows whether its previous purchases (if any) were successful. The case of $\lfloor \rho K \rfloor = 0$ was covered in the text but here $\lfloor \rho K \rfloor$ can be any nonnegative integer.

Having the option to buy more than one idea in the first subperiod means that the expression for $Z(K, B)$ given in (2) is no longer valid. However, conditional on having the right Z function, the expressions for V^D, V^R and W given in (15), (16) and (18) remain valid. Thus, the task in this Appendix is to describe the equations that yield $Z(K, B)$.

To do this, we divide the first subperiod into $\lfloor \rho K \rfloor$ stages and proceed sequentially. Let $j \in \{0, 1, 2, \ldots, \lfloor \rho K \rfloor\}$ indicate the stage number. At the start of stage j, the firm is presented with its $(j+1)^{th}$ opportunity to purchase a new idea and must decide whether to do so knowing only how many of the previous j purchase opportunities resulted in a new variety. At stage j the number of varieties owned by the firm is some integer in $\{K, K+1, \ldots, K+j\}$, depending on how many of the past purchase opportunities resulted in a new variety.

To solve this sequence of decision problems, we proceed by backward induction.

- For $j = \lfloor \rho K \rfloor$ and $K_j \in \{K, K+1, \dots \lfloor \rho K \rfloor\}$ let $s_j^*(K_j, B; \lfloor \rho K)\rfloor)$ solve $s[W(K_j+1, B) - W(K_j, B)] = \sigma W(1,0)$. The firm purchases the idea iff $K_j < K_{\max}$ and $s \geq s_j^*(K_j, B; \lfloor \rho K)\rfloor)$. Define

$$Z_j(K_j, B; \lfloor \rho K)\rfloor)) = \tag{38}$$
$$\begin{cases} W(K,B) & \text{if } K_j = K_{\max} \\ [\rho K - \lfloor \rho K \rfloor] \int_0^1 \mathbb{1}_{\{s \geq s_j^*(K_j, B; \lfloor \rho K \rfloor)\}} [sW(K_j+1, B) + (1-s)W(K_j, B) & \\ -\sigma W(1,0)] \, dF(s) \; + [1 - [\rho K - \lfloor \rho K \rfloor] \int_0^1 \mathbb{1}_{\{s \geq s_j^*(K, B; \lfloor \rho K \rfloor)\}} dF(s)] W(K,B) & \text{if } K < K_{\max}. \end{cases}$$

- For $j \in \{0, \dots, \lfloor \rho K \rfloor - 1\}$ and $K_j \in \{K, K+1, \dots, K+j\}$, let $s_j^*(K_j, B; \lfloor \rho K)\rfloor)$ solve $s[Z_{j+1}(K_j+1, B; \lfloor \rho K \rfloor) - Z_{j+1}(K_j, B; \lfloor \rho K \rfloor)] = \sigma W(1,0)$. The firm purchases the idea iff $K_j < K_{\max}$ and $s \geq s_j^*(K_j, B; \lfloor \rho K)\rfloor)$.

  Define

$$Z_j(K_j, B; \lfloor \rho K)\rfloor)) = \tag{39}$$
$$\begin{cases} Z_{j+1}(K_j, B; \lfloor \rho K \rfloor) & \text{if } K_j = K_{\max} \\ \int_0^1 \mathbb{1}_{\{s \geq s_j^*(K_j, B; \lfloor \rho K \rfloor)\}} [sZ_{j+1}(K_j+1, B; \lfloor \rho K \rfloor) + (1-s)Z_{j+1}(K_j, B; \lfloor \rho K \rfloor) & \\ -\sigma W(1,0)] \, dF(s) \; + [1 - \int_0^1 \mathbb{1}_{\{s \geq s_j^*(K, B; \lfloor \rho K \rfloor)\}} dF(s)] Z_{j+1}(K_j, B; \lfloor \rho K \rfloor) & \text{if } K < K_{\max}. \end{cases}$$

Then,

$$Z(K,B) = Z_0(K, B; \lfloor \rho K \rfloor). \tag{40}$$

Appendix C A Default Cost Model

The goal of this section is to show that the constraint on default probabilities is not necessary for the main results. We study an environment in which lenders incur a fixed cost $\Delta > 0$ in the event of default. In this model, even if θ is set equal to 1, there is a difference in the access to external finance for small and large firms.

Table 9

Statistic	Base Model	Default Cost Model
Probability of default (%)	1.0	0.02
Empl Share of Entrants (%)	2.69	2.66
Survival rate of 1-yr-old firms	0.85	0.85
Equilibrium value of ρ	0.27	0.27

All equations are as in the main text, except that the equilibrium condition for the price of debt is now

$$\begin{aligned} & q(K', B')(1+r) \qquad (41) \\ &= \left[\rho K' \int s \mathbb{1}_{\{s \geq s^*(K',B')\}} dF(s)\right] \mathbb{E}_{(K''|K'+1)} \left[[1 - D(K'', B')] + D(K'', B') \frac{\bar{B}(K'') - \Delta}{B'}\right] + \\ & \quad \left[1 - \rho K' \int s \mathbb{1}_{\{s \geq s^*(K',B')\}} dF(s)\right] \mathbb{E}_{(K''|K')} \left[[1 - D(K'', B')] + D(K'', B') \frac{\bar{B}(K'') - \Delta}{B'}\right], \end{aligned}$$

where, as before, $\bar{B}(K'') = \pi K'' + G(K'')$.

In a competitive world, creditors have to be compensated for the loss of Δ in the event of default, which means that firms must pay a higher interest rate in the state of the world in which they do not default. Since default is more likely for smaller firms for any level of debt, they must pay a higher interest rate (obtain a lower price) than larger firms for the same level of debt.[42]

Remarkably, this model is as capable as the model in the main text in accounting for the leverage-size relationship: Other model statistics barely change, except that the average default probability is much lower. The fixed cost model has difficulty accounting simultaneously for both the default rate and the response of leverage to firm size, which is a consequence of the low leverage of small firms relative to the base model. However, the effect of the declining path of interest rate is almost identical to that of the main model: The entrant's employment share falls from 2.66 percent to 1.83 percent and the entry rate falls from 8.22 percent to 6.91 percent.

[42] Furthermore, for the same default risk, the fixed cost makes the interest rate on smaller loans higher.

Appendix D

Lemma 1. *Denote the probability of K'' varieties surviving out of $K' \geq K''$ varieties by $h(K'', K')$ and the probability of at most K'' varieties surviving by $H(K'', K')$. Then, $H(K'', K'+1) < H(K'', K')$.*

Proof. As in the text, denote the probability of variety surviving by $(1-\phi)$. Then,

$$\begin{aligned} H(K'', K'+1) &= (1-\phi)H(K''-1, K') + \phi H(K'', K') \\ &= (1-\phi)\sum_{k=0}^{K''-1} h(k, K') + \phi \sum_{k=0}^{K''} h(k, K') \\ &= \sum_{k=0}^{K''-1} h(k, K') + \phi h(K'', K) \\ &= \sum_{k=0}^{K''} h(k, K') - (1-\phi)h(K'', K') \\ &= H(K'', K') - (1-\phi)h(K'', K') \\ &< H(K'', K'). \end{aligned}$$

The first equality is clearly true; the second follows from the definitions of H and h; the third by cancellation of terms; the fourth by adding and subtracting $(1-\phi)h(K'', K)$; and the fifth from definition of $H(K'', K')$. □

Proposition 4. *If $\pi\widetilde{K} + G^*(\widetilde{K})$ is increasing in $\widetilde{K}$, $d^*(K', B')$, is decreasing in K'.*

Proof. We have

$$\begin{aligned} d^*(K', B') = p^*(K', B') \cdot \Pr\left[\pi K'' + G^*(K'') \leq B' \; |K'+1\right] \; + \\ [1 - p^*(K', B')] \cdot \Pr\left[\pi K'' + G^*(K'') \leq B' \; |K'\right]. \end{aligned}$$

Since $\pi K + G^*(K)$ is increasing in K the event in $[\cdot]$ is a tail event, namely, the event that the number of surviving varieties falls below some debt-determined threshold $\bar{K}(B')$.

By Lemma 1, $\Pr[K'' < \bar{K}(\ell)|K'+1] \leq \Pr[K'' < \bar{K}(\ell)|K']$ for any K' and $K'+1 \in \mathbb{K}$. Therefore the two Pr terms must decline with K'. However, an increase in K' given B' also changes $p^*(B', K')$

and this effect must be taken into account. But observe that for any pair $\{p_0^*, p_1^*\}$, Lemma 1 implies

$$p_0^* \Pr[K'' < \bar{K}(B')|K'] + (1 - p_0^*) \Pr[K'' < \bar{K}(B')|K' + 1] \geq \Pr[K'' < \bar{K}(B')|K' + 1]$$

and

$$\Pr[K'' < \bar{K}(B')|K' + 1] \geq p_1^* \Pr[K'' < \bar{K}(B')|K' + 2] + (1 - p_1^*) \Pr[K'' < \bar{K}(B')|K' + 1].$$

Therefore, regardless of how $p^*(K', B')$ changes with K', $d(K', B') \geq d(K' + 1, B')$. Furthermore, the inequality is strict if at least one p_k^* is in $(0, 1)$. □

Proposition 5. *If $\pi\widetilde{K} + G^*(\widetilde{K})$ is increasing in $\widetilde{K}$ and $p^*(K', B')$ is decreasing in B', $d^*(K', B')$ is increasing in B'.*

Proof. Consider two leverage levels B' and $B' + \Delta$, $\Delta > 0$. Denote the associated $p^*(K', B')$ and $p^*(K', B' + \Delta)$ by p and p_Δ. By assumption, $p \geq p_\Delta$. By Lemma 1,

$$p \Pr[K'' < \bar{K}(B')|K' + 1] + (1 - p) \Pr[K'' < \bar{K}(B')|K'] \leq$$
$$p_\Delta \Pr[K'' < \bar{K}(B')|K' + 1] + (1 - p_\Delta) \Pr[K'' < \bar{K}(B')|K'],$$

and, since $\bar{K}(B')$ — the threshold below which default is triggered — is clearly increasing in B'

$$p_\Delta \Pr[K'' < \bar{K}(B')|K' + 1] + (1 - p_\Delta) \Pr[K'' < \bar{K}(B')|K'] \leq$$
$$p_\Delta \Pr[K'' < \bar{K}(B' + \Delta)|K' + 1] + (1 - p_\Delta) \Pr[K'' < \bar{K}(B' + \Delta)|K'].$$

Therefore, $d^*(K', B')$ is increasing in B'. □